117. An Ergodic Theorem Associated with Harmonic Integrals.

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Let R be an n-dimensional $(n \geqq 2)$, infinitely differentiable, orientable Riemann space, closed or open. We consider the totality H' of the infinitely differentiable exterior differential p-forms α on R which vanish outside compact sets of R. Let $d\alpha$ and $\delta\alpha = (-1)^{np+n}(d\alpha^*)^*$ be the exterior differential and codifferential of $\alpha \in H'$, so that we have $(d\alpha, \beta) = -(\alpha, \delta\beta)$. Here $(\alpha, \beta) = \int_R \alpha\beta^*$ and β^* is the adjoint form of β. Let H be the Hilbert space obtained as the completion of the pre-Hilbert space H', metrized by the norm $\|\alpha\| = (\alpha, \alpha)^{1/2}$. We denote by H'' the linear subspace of H consisting of the totality of the infinitely differentiable p-forms α such that α and $\varDelta\alpha$, $\varDelta = d\delta + \delta d$ (the Laplacian), both belong to H. Surely we have $H' \subseteqq H'' \subseteqq H$.

Suggested by an interesting paper by A. N. Milgram and P. C. Rosenbloom[1], we will prove the following ergodic theorem as a stochastic interpretation and proof of Hodge's theory of harmonic integrals[2].

The ergodic theorem. Let us consider $\varDelta$ as a symmetric, non-positive definite operator defined on $H' \subseteqq H$ with values in $H' \subseteqq H$. Let $\tilde{\varDelta}$ be the Friedrichs-Freudenthal's[3] self-adjoint, non-positive definite extension of $\varDelta$; $\tilde{\varDelta}$ is the adjoint $\varDelta'$ of $\varDelta$ restricted to the domain $D(\tilde{\varDelta})$ which is the intersection of the domain $D(\varDelta')$ of $\varDelta'$ with the completion H_1 of H' by the new norm $\|\alpha\|_1 = (\alpha - \varDelta\alpha, \alpha)^{1/2}$. If we consider the diffusion equation

$$(1) \qquad \frac{\partial T_t\alpha}{\partial t} = \text{strong} \lim_{\delta \to 0} \frac{T_{t+\delta}\alpha - T_t\alpha}{\delta} = \tilde{\varDelta} T_t\alpha, \ t \geqq 0,$$

with the initial condition

1) Harmonic forms and heat conduction, Proc. Nat. Acad. Sci., **37** (1951), 180–184.

2) W. V. D. Hodge: The theory and applications of harmonic integrals, Cambridge (1941). Cf. P. Bidal-G. de Rham: Le formes différentielles harmoniques, Comm. Math. Helv., **19** (1946), 1–49. K. Kodaira: Harmonic fields in Riemannian manifolds, Ann. of Math., **50** (1949), 581–665, G. de Rham–K. Kodaira: Harmonic integrals, Princeton (1950).

3) K. Friedrichs: Spektraltheorie halbbeschränkter Operatoren, Math. Ann., 109 (1934), 465–487. H. Freudenthal: Über Friedrichsche Fortsetzung halbbeschränkter Hermitescher Operatoren, Proc. Acad. Amsterdam, **39** (1936), 832–833.

(2) strong $\lim_{t\to 0} T_t\alpha = \alpha \in D(\tilde{\Delta})$,

then there exists $T_\infty\alpha \in H''$ such that

(3) strong $\lim_{t\to\infty} T_t\alpha = T_\infty\alpha$ and

$$\Delta T_\infty\alpha = 0,\ (\alpha - T_\infty\alpha, \beta) = 0 \text{ for any } \beta\in H'' \text{ with } \Delta\beta = 0.$$

Supplement. i) If R is closed, then we may take, in (1), $\tilde{\Delta}$ as the smallest closed extension $(\Delta')'$ of Δ. Moreover, when $\alpha \in H''$,

(4) $\int_0^\infty T_t(\alpha - T_\infty\alpha)dt$ may be considered to belong to H'' and

$$T_\infty\alpha - \alpha = \Delta\int_0^\infty T_t(\alpha - T_\infty\alpha)dt.$$

ii) When R is the euclidean n-space, we may take, in (1), $\tilde{\Delta}$ as the smallest closed extension $(\Delta')'$ of Δ.

Proof of the ergodic theorem. Since Δ is self-adjoint with its spectra lying on the half line $(-\infty, 0)$, we may apply the semi-group theory[4]. Thus there exists a one-parameter semi-group of linear operators $T_t\ (t \geqq 0)$:

(5) $T_t\alpha = \text{strong} \lim_{m\to\infty} (I - m^{-1}t\tilde{\Delta})^{-m}\alpha,\ \alpha \in H,$

(I = the identity operator)

such that

(6) $T_0 = I,\ T_tT_s = T_{t+s}$, strong $\lim_{t\to 0} T_t\alpha = \alpha$ for $\alpha\in H$, $\| T_t \| \leqq 1$

and

(1)′ $\frac{\partial T_t\alpha}{\partial t} = \text{strong} \lim_{\delta\to 0} \frac{T_{t+\delta} - T_t}{\delta}\alpha$ exists only for $\alpha \in D(\tilde{\Delta})$ and $= \tilde{\Delta}T_t\alpha$,

(1)′′ $T_t\alpha - \alpha = \int_0^t \tilde{\Delta}T_t\alpha dt$ for $\alpha \in D(\tilde{\Delta})$.

Let

(7) $\tilde{\Delta} = \int_{-\infty}^0 \lambda dE_\lambda$

be the spectral resolution of $\tilde{\Delta}$. Then, by (5), we have

(8) $T_t\alpha = \int_{-\infty}^0 \exp(\lambda t) dE_\lambda\alpha,\ \alpha\in H.$

Hence, by

$$\| T_t\alpha - T_{t'}\alpha \|^2 = \int_{-\infty}^0 |\exp(\lambda t) - \exp(\lambda t')|^2 d\| E_\lambda\alpha \|^2,$$

we easily see that

4) E. Hille Functional analysis and semi-groups, New York (1948). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, J. Math. Soc. Japan, **1** (1948), 15–21.

5) L. Schwartz: Théorie des distributions, Paris (1950).

(9) strong $\lim_{t\to\infty} T_t\alpha = T_\infty\alpha$ exists and $T_tT_\infty\alpha = T_\infty\alpha$ for $\alpha \in H$.

Therefore, by (1)$'$, we have

(10) $\tilde{\Delta}T_\infty\alpha = 0$ for $\alpha \in H$.

Consider the "distribution"[5] $S(\gamma) = (\gamma, T_\infty\alpha)$, $\gamma \in H'$. Then, by (10), S satisfies the differential equation in the sense of the "distribution":

$$\Delta S = 0 .$$

Since Δ is "elliptic", there exists[6] infinitely differentiable β such that $\Delta\beta = 0$ and $S(\gamma) = (\gamma, \beta)$ for every $\gamma \in H'$. Hence we may consider that $T_\infty\alpha$ is in H'' and

(10)$'$ $\Delta T_\infty\alpha = 0$.

Thus we see that

(11) $T_\infty = E_0 - E_{0-0}$, viz. T_∞ is the projection operator upon the closed linear subspace spanned by the solutions $\beta \in H''$ of $\Delta\beta = 0$.

Hence, starting from any $\alpha \in H$, we may obtain its harmonic part $T_\infty\alpha$ by a stochastic procedure (9). For we have

(12) $\Delta\beta = 0\,(\beta \in H'')$ implies $T_\infty\beta = \beta$ and hence

$$(\alpha - T_\infty\alpha, \beta) = (\alpha, \beta) - (T_\infty\alpha, \beta) = (\alpha, \beta) - (\alpha, T_\infty\beta)$$
$$= (\alpha, \beta) - (\alpha, \beta) = 0 .$$

The proof of the supplement. i) We have

(13) $\| (I-\Delta)\alpha \|^2 \geqq \| \alpha \|^2 - (\Delta\alpha, \alpha) \geqq \| \alpha \|^2$.

Moreover, the range $\{\beta ;\ \beta = (I-\Delta)\alpha,\ \alpha,\ \alpha \in H'\}$ is strongly dense in H. For, if otherwise, there would exist $\tilde{\gamma} \in H$ such that $\tilde{\gamma} \neq 0$ and $((I-\Delta)\alpha, \tilde{\gamma}) = 0$ for every $\gamma \in H'$. By the "ellipticity" of the operator $(I-\Delta)$, we see that $\tilde{\gamma}$ may be considered to be infinitely differentiable. Thus, by (13), $\tilde{\gamma}$ must equal to 0, contrary to $\tilde{\gamma} \neq 0$. Hence the resolvent $(I-\tilde{\Delta})^{-1}$ exists as a bounded self-adjoint operator defined on H. We have, from (13),

(14) $\| (I-\Delta)\alpha \| \leqq 1$ implies $\| \alpha \| \leqq 1$ and $0 \leqq -(\Delta\alpha, \alpha) \leqq 1$.

By virtue of the compactness of R, we see, by extending F. Rellich's argument[7], that the resolvent $(I - \tilde{\Delta})^{-1}$ is completely continuous. Hence the spectra of form a discrete set which accumulates only at $-\infty$. Hence

(7)$'$ $\tilde{\Delta} = \sum_{i=1}^{\infty} \lambda_i P_i$, $O = \lambda_1 > \lambda_2 > \cdots$, $\lim_{i\to\infty} \lambda_i = -\infty$, where P_i is the projection on the closed linear subspace spanned by the solutions $\beta \in H''$ of $\Delta\beta = \lambda_i\beta_i$.

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6) L. Schwartz: ibid.

7) Ein Satz über mittlere Konvergenz, Göttingen Nachrichten (1930), 15-21.

Here we have again made use of the "ellipticity" of the operator $(\lambda_i I - \Delta)$. Thus we have

$$(8)' \qquad T_t\alpha = \sum_{i=1}^{\infty} \exp(\lambda_i t) P_i \alpha, \quad \alpha \in H.$$

By $P_1 = T_\infty$, $P_i P_j = 0 \ (i \neq j)$, we have

$$T_t(\alpha - T_\infty \alpha) = \sum_{i=2}^{\infty} \exp(\lambda_i t) P_i \alpha .$$

Hence the integral (in the sense of S. Bochner)

$$\int_0^t T_t(\alpha - T_\infty \alpha) dt$$

converges strongly, when $t \uparrow \infty$, to an element $\int_0^\infty T_t(\alpha - T_\infty \alpha) dt \in H$. Thus, by $(1)''$ and the closure of the operator $\tilde{\Delta}$, we obtain

$$(15) \qquad T_\infty \alpha - \alpha = \tilde{\Delta} \int_0^\infty T_t(\alpha - T_\infty \alpha) dt \quad \text{for} \quad \alpha \in H''.$$

Since $T_\infty \alpha - \alpha$ is infinitely differentiable and since the operator Δ is "elliptic", $\int_0^\infty T_t(\alpha - T_\infty \alpha) dt$ may be considered to be infinitely differentiable and so we have (4).

ii) It would be sufficient, as above, to prove that the range $\{\beta;\ \beta = (I - m^{-1}\Delta)\alpha,\ \alpha \in H'\}$ is, for $m > 0$, strongly dense in H. Let us assume the contrary. Then there must exist $\tilde{\gamma} \in H$ such that $\tilde{\gamma} \neq 0$ and $((I - m^{-1}\Delta)\alpha, \tilde{\gamma}) = 0$ for every $\alpha \in H'$. By the "ellipticity" of the operator $(I - m^{-1}\Delta)$, there must exist infinitely differentiable γ which satisfies $(I - m^{-1}\Delta)\gamma = 0$ and $(\alpha, \tilde{\gamma}) = (\alpha, \gamma)$ for every $\alpha \in H'$. Such γ is, in the case of the ordinary Laplacian Δ in the euclidean n-space $(n \geqq 2)$[8], identically zero, contrary to $\tilde{\gamma} \neq 0$.[9]

8) See K. Yosida: A theorem of Liouville's type for meson equation, Proc. Japan Acad., **27** (1951), 214-215.

9) At this juncture, the author wishes to express his hearty thanks to J. Igusa who kindly pointed out that, in the general case of open space R, the smallest closed extension$(\Delta')'$ may have positive spectra. The original manuscript is revised according to his criticism.