115. A Remark on the Efficiency of the Designs of Weighing Experiments.

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The problem of choosing the best design for improving efficiencies of weighing experiments was first discussed by H. Hitelling[1] in 1944. H. Hotelling succeeded in formulating the problem in an elegant form; i.e., as the problem of finding the best linear unbiased estimates[2] of the true weights of the objects being weighed from the readings of the scale. Subsequently in 1945, K. Kishen[3] defined the efficiency of the design by means of the mean variance of the best linear unbiased estimates, and he examined the possibilities of the completely orthogonal designs in various cases.

We shall denote true weights of the p objects to be weighed by a column vector

$$\mathfrak{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix}.$$

For the sake of simplicity, we shall consider the case where there is no bias in reading the scale, and any of the p objects are weighed just N times either on the left pan or on the right pan. The readings y_α, $\alpha = 1, \dots, N$, of the scale are considered as mutually independent random variables and having a constant variance σ^2. The mean value vector are expressed as follows;

$$\begin{pmatrix} E(y_1) \\ E(y_2) \\ \vdots \\ \vdots \\ E(y_N) \end{pmatrix} = \begin{pmatrix} x_{11} & x_{21} \dots x_{p1} \\ x_{12} & x_{22} \dots x_{p2} \\ \dots & \dots \\ \dots & \dots \\ x_{1N} & x_{2N} \dots x_{pN} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_p \end{pmatrix},$$

or in short.

$$[E(y)] = X \cdot \mathfrak{b}.$$

The value of $x_{i\alpha}$ is either $+1$ or -1 according as the i-th object of which the true weight is b_i is placed on the left pan or on the right pan.

In such cases as above, the best linear unbiased estimates $\hat{\mathfrak{b}}$ of $\mathfrak{b}$ where

$$\hat{\mathfrak{b}} = \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \hat{b}_p \end{pmatrix}$$

will be easily obtained by means of the famous theorem due to A. Merkoff[4].

Let it be

$$a_{ij} = \sum_{\alpha=1}^{N} x_{i\alpha} x_{j\alpha}, \quad z_i = \sum_{\alpha=1}^{N} x_{i\alpha} y_\alpha, \quad i, j = 1, \dots, p,$$

and

$$A = (a_{ij}) = X'X,$$
$$A^{-1} = (a^{ij}),$$

then we have

$$\hat{\mathfrak{b}} = A^{-1} \cdot \mathfrak{z},$$

where

$$\mathfrak{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ \mathfrak{z}_p \end{pmatrix}.$$

The variance and coveriance matrix of the joint distribution of $\hat{\mathfrak{b}}$ is seen to be $\sigma^2 \cdot A^{-1}$.

In view of the mutual dependence of $\hat{b}_1, \dots, \hat{b}_p$, the author considers K. Kishen's definition[5] of the efficiency, i.e.,

$$v_m = \frac{\sigma^2}{N} \cdot \frac{N \sum_i a^{ii}}{p}$$

as an insufficient one, and he shall propose a new definition in the following.

As is well known, the ellipsoid of concentration[6] of the joint distribution of $\hat{\mathfrak{b}}$ is

$$\sum_i \sum_j a^{ij} (\hat{b}_i - b_i)(\hat{b}_j - b_j) = p + 2.$$

If the frequency function of the reading y_α of the scale are known, then we can define the efficiency of the design as in the usual way[ ].

Otherwise, we proceed as follows: The volume of the ellipsoid of concentration is proportional to

$$|A|^{-\frac{1}{2}}$$

where $|A|$ denotes the determinant of the matrix A.

The matrix A is symmetric and positive-definite or semi-definit by its definition. We shall consider the case where A is positiv-definite, in other words, the p columns of the matrix X are linearly independent. Then, we have

$$|A| \leqq a_{11}a_{22}\dots a_{pp} = N^p.$$

The sign of equality holds when and only when the p columns of X are mutually orthogonal.

We shall define the efficiency of a design as

$$E = \frac{|A|}{N^p}.$$

In the case[9], for instance,

$$N = 2^m+1,\ p \leqq 2^m \qquad \text{and no bias,}$$

where there is no completely orthogonal design, K. Kishen proposed one design for which

$$A = \begin{pmatrix} N\ 1 \dots\dots\dots 1 \\ 1\ N \dots\dots\dots 1 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ 1\ 1 \dots\dots\dots N \end{pmatrix}$$

as probably the most efficient one. The efficiency of this design by our new definition is seen to be

$$E = \left(1+\frac{p-1}{N}\right)\left(1-\frac{1}{N}\right)^{p-1}.$$

In particular if $p = 2^m = N-1$, there

$$E = \left(2-\frac{2}{N}\right)\left(1-\frac{1}{N}\right)^{N-2},$$

whence it follows that for indefinitely large N

$$\lim_{N\to\infty} E = \frac{2}{e} \fallingdotseq 0.732.$$

References.

1) Hotelling, H., Some improvements in weighing and other experimental techniques, Ann. Math. Stat., Vol. 15, No. 3 (1944), pp. 297-306.

2) David, F.N. and Neyman, J., Extension of the Markoff theorem on least squares, Stat. Res. Mem. Vol. 2 (1937), pp. 105-116.

3) Kishen, K., On the design of experiments for weighing and making other types of measurements, Ann. Math. Stat. Vol. 16, No. 3 (1945), pp. 294-300.

4) David, F.N. and Neyman, J., loc. cit., p. 107.

5) Kishen, K., loc. cit., p. 295.

6) Cramér, H., Mathematical Methods of Statistics, Princeton, 1946, p. 300.

7) Cramér, H., loc. cit., Chap. 32.

8) Kishen, K., loc. cit., p. 297.