138. *On the Simple Extension of a Space with Respect to a Uniformity. IV.*

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The purpose of the present note is to show that any regular T_1-space containing a regular T_1-space R as a dense subset can be obtained by constructing the simple extension of R with respect to some regular uniformity[1], and to discuss some other extensions related to the simple extensions.

§ 1. Regular uniformity.

Theorem 1. *Let $\{\mathfrak{U}_\alpha; \alpha\in\Omega\}$ be a regular uniformity of a space R agreeing with the topology. Then the simple extension R^* of R with respect to $\{\mathfrak{U}_\alpha\}$ is characterized as a space S with the following properties:*

(1) *S contains R as a subspace.*

(2) *$\{S-\overline{R-G}\,;\ G$ open in $R\}$ is a basis of open sets for S.*

(3) *Each point of $S-R$ is closed.*

(4) *$\mathfrak{V}_\alpha=\{S-\overline{R-U}\,;\ U\in\mathfrak{U}_\alpha\}$ is an open covering of S.*

(5) *$\{S(x,\mathfrak{V}_\alpha)\,;\alpha\in\Omega\}$ is a basis of neighbourhoods at each point x of $S-R$.*

(6) *S is complete with respect to the uniformity $\{\mathfrak{V}_\alpha\,;\alpha\in\Omega\}$.*

Here the bar indicates the closure operation in S.

Proof. It is proved by I, Theorem 9 that R^* has the properties (1)-(6). Conversely, let S be a space with the properties (1)-(6). For any point x of $S-R$, $\{S(x,\mathfrak{V}_\alpha)\cdot R\,;\alpha\in\Omega\}$ is a Cauchy family with respect to $\{\mathfrak{U}_\alpha\}$ because of the regularity of $\{\mathfrak{U}_\alpha\}$, and hence for any $\alpha\in\Omega$ there exists $\beta,\gamma\in\Omega$ and $U_\alpha\in\mathfrak{U}_\alpha$ such that $S(S(x,\mathfrak{V}_\beta)\cdot R, \mathfrak{U}_\gamma)\subset U_\alpha$. Hence we have $\overline{S(x,\mathfrak{V}_\beta)\cdot R}\subset S(S(x,\mathfrak{V}_\beta)\cdot R,\mathfrak{V}_\gamma)\subset S-\overline{R-U_\alpha}$ $\subset S(x,\mathfrak{V}_\alpha)$.

Since $\{\mathfrak{V}_\alpha\}$ agrees with the topology of S, $\{S(x,\mathfrak{V}_\alpha)\cdot R\,;\alpha\in\Omega\}$ is a vanishing Cauchy family of R with respect to $\{\mathfrak{U}_\alpha\}$ such that $x=\Pi\,\overline{S(x,\mathfrak{V}_\alpha)\cdot R}$. Therefore Theorem 1 follows immediately from II, Theorem 1.

Theorem 2. *Let R be a regular T-space, and let S be any regular T-space such that S contains R as a dense subspace and each point of S-R is closed. Then there exists a homeomorphism φ of S*

1) K. Morita: On the simple extension of a space with respect to a uniformity. I, II, III, Proc., **27** (1951), 65-72; 130-137; 166-171. These notes shall be cited with I, II, III respectively. We make use of the same terminologies and notations as in these notes.

onto the simple extension R^* *of* R *with respect to a regular T-uniformity* $\{\mathfrak{U}_\alpha; \alpha\in\Omega\}$, *agreeing with the topology, such that* φ *leaves each point of* R *invariant. Here* $\{\mathfrak{U}_\alpha; \alpha\in\Omega\}$ *can be obtained as* $\mathfrak{U}_\alpha=\{W\cdot R; W\in\mathfrak{W}_\alpha\}$ *from any regular T-uniformity* $\{\mathfrak{W}_\alpha; \alpha\in\Omega\}$ *of* S, *agreeing with the topology, such that* S *is complete with respect to* $\{\mathfrak{W}_\alpha; \alpha\in\Omega\}$.

Remark 1. As $\{\mathfrak{W}_\alpha; \alpha\in\Omega\}$ in Theorem 2 we may take the family of all open coverings of S by I, Theorems 1 and 8.

Remark 2. It is to be noted that there are completely regular T_1-spaces R and S such tnat S is not homeomorphic to the simple extension of R with respect to any *completely regular* T-uniformity although S contains R as a dense subspace.

Proof of Theorem 2. Let $\{\mathfrak{W}_\alpha; \alpha\in\Omega\}$ be a regular T-uniformity of S, agreeing with the topology, such that S is complete with respect to $\{\mathfrak{W}_\alpha\}$. If we put $\mathfrak{U}_\alpha=\{W\cdot R; W\in\mathfrak{W}_\alpha\}$ for each $\alpha\in\Omega$, then $\{\mathfrak{U}_\alpha; \alpha\in\Omega\}$ is a regular T-uniformity of R agreeing with the topology. We shall remark that for any open set G of S we have

$$G\subset S-\overline{R-GR}\subset \overline{G},$$

where the bar indicates the closure operation in S. The first relation is evident, since $R-G\cdot R=(S-G)\cdot R$, and we have $S=\overline{R}=\overline{R-G\cdot R}+\overline{G}$ by the additivity of the closure operation.

Therefore the family $\{S-\overline{R-G}; G \text{ open in } R\}$ is a basis of open sets for S, and, if we put $\mathfrak{V}_\alpha=\{S-\overline{R-U}; U\in\mathfrak{U}_\alpha\}$, $\mathfrak{V}_\alpha$ is an open covering of S and the uniformity $\{\mathfrak{V}_\alpha; \alpha\in\Omega\}$ is equivalent to $\{\mathfrak{W}_\alpha; \alpha\in\Omega\}$. This completes our proof by Theorem 1.

Theorem 3. *Let* R *be a completely regular T-space, and let* S, T *be the simple extension of* R *with respect to completely regular T-uniformities* $\{\mathfrak{U}_\alpha\}$, $\{\mathfrak{V}_\lambda\}$ *respectively, where both uniformities agree with the topology and consist of finite open coverings (whence follows that* S, T *are bicompact). In order that there exist a continuous mapping* f *from* S *onto* T *such that* $f(S-R)=T-R$ *and* $f(x)=x$ *for every point* x *of* R, *it is necessary and sufficient that for any* $\mathfrak{V}_\lambda$ *of* $\{\mathfrak{V}_\lambda\}$ *there exist some* $\mathfrak{U}_\alpha$ *of* $\{\mathfrak{U}_\alpha\}$ *which is a refinement of* $\mathfrak{V}_\lambda$.

Proof. The sufficiency follows from II, Theorem 3 (cf. also the proof of III, Theorem 5). We shall prove the necessity. Let $\mathfrak{V}_\lambda=\{V_1, \cdots, V_m\}$ be any covering of $\{\mathfrak{V}_\lambda\}$. Then $\{T-\overline{R-V_i}; i=1, 2, \cdots, m\}$ is an open covering of T, and hence $\{S-f^{-1}(\overline{R-V_i}); i=1, \cdots, m\}$ is also an open covering of S. Since $\overline{R-V_i}\subset f^{-1}(\overline{R-V_i})$, $\{S-\overline{R-V_i}; i=1, \cdots, m\}$ is an open covering of S. The remark at the end of II, § 2 shows that thore exists a refinement $\mathfrak{U}_\alpha\in\{\mathfrak{U}_\alpha\}$ of $\mathfrak{V}_\lambda$.

§ 2. **Strong agreement with the topology.** A uniformity $\{\mathfrak{U}_\alpha; \alpha\in\Omega\}$ of a space R is said to *agree strongly with the topology*, if $\{S(S(x,\mathfrak{U}_\alpha), \mathfrak{U}_\beta); \alpha, \beta\in\Omega\}$ is a basis of neighbourhoods at each point x of R. Then we have

Lemma 1. *In case* $\{\mathfrak{U}_\alpha\}$ *is a regular uniformity,* $\{\mathfrak{U}_\alpha\}$ *agrees strongly with the topology if and only if it agrees with the topology in the ordinary sense.*

As an application of this notion we mention

Theorem 4. *Let R be a T_1-space. In order that R be metrizable it is necessary and sufficient that there exist a uniformity* $\{\mathfrak{U}_n;\, n=1, 2, \cdots\}$ *which consists of a countable number of open coverings and agrees strongly with the topology.*

Indeed Theorem 4 is shown to be a reformulation of a result due to A.H. Frink [2].

§ 3. **The regular extension with respect to a uniformity.** Let $\{\mathfrak{U}_\alpha;\, \alpha\in\Omega\}$ be a uniformity of a space R. A family $\{X_\lambda\}$ of subsets of R is called a Cauchy family of rank $n(n\geqq 1)$ if it has the finite intersection property and for any $\alpha\in\Omega$ there exist $\beta_1, \cdots, \beta_n\in\Omega$, $U_\alpha\in\mathfrak{U}_\alpha$, $X_\lambda\{X_\lambda\}$ such that

$$S(S(\cdots(S(X_\lambda, \mathfrak{U}_{\beta_1}), \cdots), \mathfrak{U}_{\beta_{n-1}}), \mathfrak{U}_{\beta_n})\subset U_\alpha .$$

Any Cauchy family of rank 2 is necessarily a Cauchy family of rank $n(n\geqq 3)$ and a Cauchy family in the ordinary sense (I, § 3). We denote hy R^+ the subspace of the simple extension R^* of R, which is made up of the points of R^*-R represented by equivalence classes of vanishing Cauchy families of rank 2 and of the points of R. This space R^+ was introduced by J. Suzuki and investigated.[3]

Lemma 2. *Each point of R^+-R is a regular point. More precisely* $\{S(S(x, \mathfrak{U}_\alpha^+), \mathfrak{U}_\beta^+);\, \alpha, \beta\in\Omega\}$ *is a basis of neighbourhoods at x of R^+-R where* $\mathfrak{U}_\alpha^+=\{U^*\cdot R^+;\, U\in\mathfrak{U}_\alpha\}$.

Proof. If $\{X_\lambda\}$ is a vanishing Cauchy family of rank 2 and belongs to x, then $S(S(X_\lambda, \mathfrak{U}_\alpha)\, \mathfrak{U}_\beta)\subset G$ implies $S(S(x, \mathfrak{U}_\alpha^+), \mathfrak{U}_\alpha^+)\subset G^+$, where $G^+=G^*\cdot R^+$. This proves Lemma 2.

Theorem 5. *If* $\{\mathfrak{U}_\alpha;\, \alpha\in\Omega\}$ *is a uniformity of a space R which agrees strongly with the topology, then R^+ is a regular space.*

In this case R^+ is called *the regular extension of R with respect to the uniformity* $\{\mathfrak{U}_\alpha\}$.

Proof. For a point x of R, $S(S(x, \mathfrak{U}_\alpha), \mathfrak{U}_\beta)\subset G$ implies $S(S(x, \mathfrak{U}_\alpha^+), \mathfrak{U}_\beta^+)\subset G^+$. This, together with Lemma 2, proves Theorem 5.

In case $\{\mathfrak{U}_\alpha\}$ is a regular uniformity R^+ coincides with R^*, as is shown by Suzuki. Hence we obtain the following theorem from Theorem 2.

Theorem 6. *Let R be a regular T-space, and let S be any re-*

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2) A. H. Frink: Distance functions and the metrization problems, Bull. Amer. Math. Soc., **43** (1937), 133-142, Theorem 4.

3) J. Suzuki: On the metrization and the completion of a space with respect to a uniformity, Proc., **27** (1951), 217-223.

gular T-space such that S contains R as a dense subset and each point of $S-R$ is closed. Then S can be obtained as the regular extension of R with respect to a T-uniformity $\{\mathfrak{U}_\alpha\}$ agreeing strongly with the topology.

§4. **The bicompact extension with respect to a uniformity.** Let $\{\mathfrak{U}_\alpha; \alpha\in\Omega\}$ be a uniformity of a space R. An ultrafilter $\{X_\lambda\}$ in R is said to be vanishing if $\Pi\bar{X}_\lambda=0$. Two ultrafilters $\{X_\lambda\}$ and $\{Y_\mu\}$ in R are said to be equivalent, if for any $X_\lambda\in\{X_\lambda\}$ and any $\alpha\in\Omega$ there exist $Y_\mu\in\{Y_\mu\}$ and $\beta\in\Omega$ such that $S(Y_\mu, \mathfrak{U}_\beta)\subset S(X_\lambda, \mathfrak{U}_\alpha)$, and conversely for any $Y_\mu\in\{Y_\mu\}$ and any $\beta\in\Omega$ there exist $X_{\lambda_0}\in\{X_\lambda\}$, $\gamma\in\Omega$ such that $S(X_{\lambda_0}, \mathfrak{U}_\gamma)\subset S(Y_\mu, \mathfrak{U}_\beta)$. This relation is clearly an equivalence relation. We consider all the equivalence classes of vanishing ultrafilters in R and denote the set of these classes by D. For any open set G of R we define the set G^b as follows: G^b consists of all the points of G and of all the points x of D such that for any ultrafilter $\{X_\lambda\}$ belonging to x there exist some $X_\lambda\in\{X_\lambda\}$ and $\alpha\in\Omega$ satisfying $S(X_\lambda, \mathfrak{U}_\alpha)\subset G$. Then we have (cf. I, §3)

Lemma 3. $G^b\cdot R=G$, $0^b=0$, $R^b=R+D$.

Lemma 4. *$G\subset H$ implies $G^b\subset H^b$.*

Lemma 5. *$G_1\cdots G_m=0$ implies $G_1^b\cdots G_m^b=0$.*

Lemma 6. *If $\{\mathfrak{U}_\alpha\}$ is a T-uniformity, then $(G\cdot H)^b=G^b\cdot H^b$.*

We take $\{G^b;$ open in $R\}$ as a basis of open sets of R^b. Then we have

Lemma 7. *The simple extension R^* of R is a subspace of R^b.*

Lemma 8. *For a point x of R^b-R, $\{S(X_\lambda, \mathfrak{U}_\alpha)^b; \lambda\in\Lambda, \alpha\in\Omega\}$ is a basis of neighbourhoods at x, where $\{X_\lambda; \lambda\in\Lambda\}$ is any ultrafilter belonging to x.*

Lemma 9. *For a vanishing ultrafilter $\{X_\lambda\}$ belonging to a point x of R^b-R we have $x\in\Pi\bar{X}_\lambda$ in R^b. In case $\{\mathfrak{U}_\alpha\}$ is completely regular we have $x=\Pi\bar{X}_\lambda$.*

Lemma 10. *If R is a T-space and $\{\mathfrak{U}_\alpha\}$ is a T-uniformity, then R^b is a T-space. Furthermore, if R is a T_0-space, so is R^b.*

Now we shall prove

Theorem 7. *If $\{\mathfrak{U}_\alpha\}$ agrees strongly with the topology of R, then R^b is bicompact.*

In this case R^b is called *the bicompact extension of R with respect to $\{\mathfrak{U}_\alpha\}$.*

Proof. Let $\{H_\lambda^b; \lambda\in\Lambda\}$ be any open covering of R^b, where H_λ are open sets of R. If we put $C_\lambda=R-H_\lambda$, then there exist $\lambda_i\in\Lambda$, $\alpha_i\in\Omega$ $(i=1, 2, \cdots, n)$ such that

$$(*)\qquad \prod_{i=1}^{n} S(C_{\lambda_i}, \mathfrak{U}_{\alpha_i})=0.$$

To prove this suppose that $\{S(C_\lambda, \mathfrak{U}_\alpha); \lambda\in\Lambda, \alpha\in\Omega\}$ has the finite intersection property. If a point x of R belongs to every $\overline{S(C_\lambda, \mathfrak{U}_\alpha)}$, then we have $S(x, \mathfrak{U}_\beta) \cdot S(C_\lambda, \mathfrak{U}_\alpha) \neq 0$ and hence $S(S(x, \mathfrak{U}_\beta), \mathfrak{U}_\alpha) \cdot C_\lambda \neq 0$, and consequently $x \in \Pi C_\lambda$ in R, which contradicts the relation that $\Pi C_\lambda \subset \Pi_\lambda \subset \bar{C}\Pi(R^b - H_\lambda^b) = 0$ in R^b. Hence if we construct an ultrafilter $\{Z_\nu\}$ containing $\{S(C_\lambda, \mathfrak{U}_\alpha)\}$, then $\{Z_\nu\}$ must be vanishing and belongs to some point z of $R^b - R$. Since $Z_\nu \cdot S(C_\lambda, \mathfrak{U}_\alpha) \neq 0$ implies $S(Z_\nu, \mathfrak{U}_\alpha) \cdot C_\lambda \neq 0$, we have $z \in \Pi \bar{C}_\lambda \subset \Pi(R^b - H_\lambda^b)$ in R^b by Lemma 8, but this is a contradiction since $\{H_\lambda^b\}$ is a covering of R^b. Thus the existence of λ_i, α_i satisfying (*) is proved.

Now let us put $A_i = R - S(C_{\lambda i}, \mathfrak{U}_{\alpha i})$, $i = 1, 2, \cdots, n$. Then it is easily shown that $\{S(A_i, \mathfrak{U}_{\alpha i})^b; i = 1, 2, \cdots, n\}$ is a covering of R^b. Since $S(A_i, \mathfrak{U}_{\alpha i}) \subset R - C_{\lambda i} = H_{\lambda i}$ the bicompactness of R^b is thus proved.

Let us put for a subset A of R and for $\alpha \in \Omega$

$$\mathfrak{M}(A, \alpha) = \{S(A, \mathfrak{U}_\alpha), R - \bar{A}\}.$$

Then we have

Lemma 11. *$\mathfrak{U}_\alpha$ is a refinement of $\mathfrak{M}(A, \alpha)$ for any subset A of R, and $S(A, \mathfrak{U}_\alpha) = S(A, \mathfrak{M}(A, \alpha))$.*

Theorema 8. *If $\{\mathfrak{U}_\alpha; \alpha \in \Omega\}$ is a uniformity agreeing strongly with the topology, so is the uniformity $\{\mathfrak{M}(A, \alpha); \alpha \in \Omega, A \subset R\}$, and the simple extension of R with respect to $\{\mathfrak{U}_\alpha\}$ is imbedded in the simple extension of R with respect to $\{\mathfrak{M}(A, \alpha)\}$, and moreover the bicompact extensions of R with respect to both uniformities are identical.*

Proof of Theorem 8 is obvious from Lemma 11.

Lemma 12. *The intersection of coverings $\mathfrak{M}(A_i, \alpha)$, $i = 1, 2, \cdots, n$ is a Δ-refinement of a covering $\{S(A_i, \mathfrak{U}_\alpha); i = 1, 2, \cdots, n\}$, where $A_1 + \cdots + A_n = R$.*

Lemma 13. *If $\mathfrak{U}_\beta$ is a Δ-refinement of $\mathfrak{U}_\alpha$, a covering $\{S(S(A, \mathfrak{U}_\beta), \mathfrak{U}_\beta), S(R - S(A, \mathfrak{U}_\beta), \mathfrak{U}_\beta)\}$ is a refinement of a covering $\mathfrak{M}(A, \alpha)$.*

Theorem 9. *If $\{\mathfrak{U}_\alpha\}$ is a completely regular uniformity of R agreeing with the topology, so also is the uniformity which consists of all the finite open coverings of the form: $\{S(A_i, \mathfrak{U}_\alpha); i = 1, 2, \cdots, n\}$ where $\alpha \in \Omega$ and $A_1 + \cdots + A_n = R$, and the simple extension of R with respect to this uniformity or the uniformity $\{\mathfrak{M}(A, \alpha)\}$ defined in Theorem 8 is identical with the bicompact extension of R with respect to $\{\mathfrak{U}_\alpha\}$.*

Proof. The first part is obvious from Lemmas 12, 13. The second part may be proved by virtue of Theorems 1 and 7 or directly.

In case $\{\mathfrak{U}_\alpha\}$ is a completely regular T-uniformity agreeing with the topology, it is easily seen that R^b coincides with the bicompactification of R recently given by P. Samuel[4]. It is an open problem whether the regular extention of R with respect to the uniformity mentioned in Theorem 8 or 9 is r-closed[5] or not, in case $\{\mathfrak{U}_\alpha\}$ is merely a T-uniformity agreeing strongly with the topology of R.

4) P. Samuel: Ultrafilters and compactification of uniform spaces, Trans. Amer. Math. Soc., **64** (1948), 100–132.
5) P. Alexandroff and H. Hopf: Topologie I, p. 90, footnote 1.