137. On Completely Additive Classes of Sets with Respect to Carathéodory's Outer Measure.

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The purpose of this paper is to investigate the relations between completely additive classes of sets with respect to Carathéodory's outer measure. This investigation has its source in an article by the author: *On the notion of measurability*[1].

1. Let X be an abstract space (an arbitrary set), and μ be an outer measure of Carathéodory on X, i.e., μ is a real valued function $\mu(A)$ defined for each subset A of X satisfying the following conditions:

i) $0 \leqq \mu(A) \leqq +\infty$. ii) If $A_1 \subset A_2$ then $\mu(A_1) \leqq \mu(A_2)$. iii) For any sequence of sets $\{A_n\}$ $(A_n \subset X)$ it holds the relation $\mu(\bigcup_{n=1}^{\infty} A_n) \leqq \sum_{n=1}^{\infty} \mu(A_n)$. iv) $\mu(0) = 0$ for the empty set 0.

We denote by $\mathfrak{C}(\mu)$ *the class of all measurable sets in the sense of Carathéodory with respect to the outer measure* μ[2]. We assume further that *there exists a sequence of sets* $\{K_n\}$ *such that* $K_n \in \mathfrak{C}(\mu)$, $K_n \subset K_{n+1}$, $\bigcup_{n=1}^{\infty} K_n = X$ *and* $\mu(K_n) < +\infty$. We call such a sequence $\{K_n\}$ a *fundamental finite series*. If $\mu(X) < +\infty$, then we can take $K_n = X$.

We say that a class of sets $\mathfrak{M}$ is *completely additive*, when $\mathfrak{M}$ satisfies the following conditions:

a) If $A_i \in \mathfrak{M}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{M}$. *b)* If $A \in \mathfrak{M}$, then $CA \in \mathfrak{M}$[3].

We say that $\mathfrak{M}$ is *finitely additive*, when in *a)* $\bigcup_{i=1}^{\infty}$ is replaced by $\bigcup_{i=1}^{2}$.

We say that $\mathfrak{M}$ is *μ-completely additive* (abbreviated *μ-c.a.*), when $\mathfrak{M}$ is completely additive and the relation $\mu(\bigcup_{i=1}^{\infty}(A_i \frown K_n)) = \sum_{i=1}^{\infty} \mu(A_i \frown K_n)$ (for all n) holds, if $A_i \in \mathfrak{M}$, $A_i \frown A_j = 0$ $(i \neq j)$, and $\{K_n\}$ is a fundamental finite series. We say that $\mathfrak{M}$ is *μ-finit ly additive* (abbreviated *μ-f.a.*), when $\mathfrak{M}$ is finitely additive and the above relation holds if $\bigcup_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ are replaced by $\bigcup_{i=1}^{2}$ and $\sum_{i=1}^{2}$ resp.

These definitions are *independent of the choice of the fundamental finite series* $\{K_n\}$ (by Lemma 4), and coincide with the ordinary one if $\mu(\bigcup A_i) < +\infty$ (by Lemma 3).

Let $\mathfrak{N}(\mu)$ be *the class of all sets* A *such that*

$$\mu(K_n \frown A) = \mu(K_n) - \mu(K_n \frown CA) \quad \textit{for all } n,$$

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1) By S. Enomoto, this proc. vol. 27, No. 5, p. 208. It will be denoted by [E].

2) A set E is said to be measurable in the sense of Carathéodory when $\mu(A) = \mu(A \frown E) + \mu(A \frown CE)$ holds for all $A \subset X$.

3) CA denotes the compliment of A: $CA = X - A$.

which is *independent of the fundamental finite series* $\{K_n\}$ (by Lemma 4).

By $\mathfrak{C}(\mu)$ we denote *the class of all sets* E *such that*

(O) $\mu(A)=\mu(A\frown E)+\mu(A\frown CE)$ *for all* $A\in\mathfrak{R}(\mu)$,

where we can assume $\mu(A)<+\infty$. We obtain easily:

Theorem 1. *It hold the relations* $\mathfrak{C}(\mu)\subset\mathfrak{C}(\mu)\subset\mathfrak{R}(\mu)$[4] *and* $\mathfrak{M}\subset\mathfrak{R}(\mu)$ *for every* μ*-c.* (*f.*) *a. class* $\mathfrak{M}$.

It will be found remarkable that $\mathfrak{C}(\mu)$, in stead of $\mathfrak{C}(\mu)$, plays a central rôle (Cf. Theorem 2 and Theorem 7).

2. Lemma 1. *Let* $\{K_n\}$ *be a fundamental finite series, then for an arbitrary set* $A\subset X$ *we have* $\lim_{n\to\infty}\mu(A\frown K_n)=\mu(A)$.

Proof: Easy. Cf. Halmos: Measure theory, §11, Theorem B.

Lemma 2. *If it hold* $A\subset\bigvee_{i=1}^{\infty}E_i$, $\mu(A)=\sum_{i=1}^{\infty}\mu(A\frown E_i)<+\infty$, $\mu(A)=\mu(A\frown F)+\mu(A\frown CF)$ *and* $\mu(A\frown E_i\frown F)+\mu(A\frown E_i\frown CF)=\mu(A\frown E_i)$ *for all* i, *then it holds:* $\mu(A\frown F)=\sum_{i=1}^{\infty}\mu(A\frown E_i\frown F)$.

Proof. $\mu(A)=\mu(A\frown F)+\mu(A\frown CF)\leqq\sum_{i=1}^{\infty}\mu(A\frown E_i\frown F)+\sum_{i=1}^{\infty}\mu(A\frown E_i\frown CF)=\sum_{i=1}^{\infty}\mu(A\frown E_i)=\mu(A)<+\infty$. Therefore $\mu(A\frown F)=\sum_{i=1}^{\infty}\mu(A\frown E_i\frown F)$.

Lemma 3. *If it holds* $\mu(\bigvee_{i=1}^{\infty}A_i)=\sum_{i=1}^{\infty}\mu(A_i)<+\infty$, *where* $A_i\frown A_j=0$ $(i\neq j)$, *then it holds for any* $H\in\mathfrak{C}(\mu)$: $\mu(\bigvee_{i=1}^{\infty}(A_i\frown H))=\sum_{i=1}^{\infty}\mu(A_i\frown H)$.

Proof, Put in Lemma 2 $A=\bigvee_{i=1}^{\infty}A_i$, $E_i=A_i$ and $F=H$.

Lemma 4. *If it holds* $\mu(\bigvee_{i=1}^{\infty}(A_i\frown K_n))=\sum_{i=1}^{\infty}\mu(A_i\frown K_n)$ *for all* n, *where* $A_i\frown A_j=0$ $(i\neq j)$, *for a fundamental finite series* $\{K_n\}$, *then it holds for any* $H\in\mathfrak{C}(\mu)$: $\mu(\bigvee_{i=1}^{\infty}(A_i\frown H))=\sum_{i=1}^{\infty}\mu(A_i\frown H)$.

Proof. By Lemma 3 and Lemma 1.

3. In the sequel we shall use for simplicity the notation $\mu_n(A)$ in stead of $\mu(A\frown K_n)$. If we prove an equality for μ_n then we shall get the equality for μ itself (by Lemma 1).

Theorem 2. *The class* $\mathfrak{C}(\mu)$ *is* μ*-c. a.*.

Proof. 1° It is clear that, if $E\in\mathfrak{C}(\mu)$ then $CE\in\mathfrak{C}(\mu)$.

2° If $E\in\mathfrak{C}(\mu)$ and $A\in\mathfrak{R}(\mu)$, then $A\frown E\in\mathfrak{R}(\mu)$. Because: $\mu(K_n)=\mu_n(A)+\mu_n(CA)=\mu_n(A\frown E)+\mu_n(A\frown CE)+\mu_n(CA)\geqq\mu_n(A\frown E)+\mu_n(C(A\frown E))\geqq\mu(K_n)$. Therefore $\mu(K_n)=\mu_n(A\frown E)+\mu_n(C(A\frown E))$.

3° If $E, F\in\mathfrak{C}(\mu)$, then $E\frown F\in\mathfrak{C}(\mu)$ and $E\smile F\in\mathfrak{C}(\mu)$. Because: For an arbitrary set $A\in\mathfrak{R}(\mu)$ such that $\mu(A)<+\infty$, we have $\mu(A)-\mu(A\frown C(E\frown F))=\mu(A\frown E)+\mu(A\frown CE)-\mu(A\frown C(E\frown F))\geqq\mu(A\frown E\frown F)+\mu(A\frown E\frown CF)+\mu(A\frown CE)-\mu(A\frown C(E\frown F)\frown E)-\mu(A\frown C(E\frown F)\frown CE)=\mu(A\frown(E\frown F))$, hence $\mu(A)\geqq\mu(A\frown(E\frown F))+\mu(A\frown C(E\frown F))\geqq\mu(A)$. Therefore $E\frown F\in\mathfrak{C}(\mu)$, and from 1° $E\smile F\in\mathfrak{C}(\mu)$.

4) Cf. the remark at the end of this paper.

4° If $A\in\mathfrak{R}(\mu)$, $E, F\in\mathfrak{E}(\mu)$ and $E\frown F=0$, then $\mu(A\frown(E\smile F))=\mu(A\frown E)+\mu(A\frown F)$. Because: Put $A\frown(E\smile F)$ in stead of A in (O).

5° If $E_i\in\mathfrak{E}(\mu)$ and $E_i\frown E_j=0$ $(i\neq j)$, then $\mu(\bigcup_{i=1}^{\infty}E_i)=\sum_{i=1}^{\infty}\mu(E_i)$. Because: By 4° $\sum_{i=1}^{m}\mu(E_i)=\mu(\bigcup_{i=1}^{m}E_i)\leqq\mu(\bigcup_{i=1}^{\infty}E_i)$, hence $\sum_{i=1}^{\infty}\mu(E_i)\leqq\mu(\bigcup_{i=1}^{\infty}E_i)\leqq\sum_{i=1}^{\infty}\mu(E_i)$.

6° If $E_i\in\mathfrak{E}(\mu)$ then $\bigcup_{i=1}^{\infty}E_i\in\mathfrak{E}(\mu)$. It suffices to prove this only when $E_i\frown E_j=0$ $(i\neq j)$. For any $A\in\mathfrak{R}(\mu)$ $(\mu(A)<+\infty)$ we have $\mu(A)-\mu(A\frown C(\bigcup_{i=1}^{\infty}E_i))\geqq\mu(A)-\mu(A\frown C(\bigcup_{i=1}^{n}E_i))=\mu(A\frown(\bigcup_{i=1}^{n}E_i))=\sum_{i=1}^{n}\mu(A\frown E_i)$ (by 4°), then $\mu(A)-\mu(A\frown C(\bigcup_{i=1}^{\infty}E_i))\geqq\sum_{i=1}^{\infty}\mu(A\frown E_i)\geqq\mu(A\frown(\bigcup_{i=1}^{\infty}E_i))$, hence $\mu(A)=\mu(A\frown(\bigcup_{i=1}^{\infty}E_i))+\mu(A\frown C(\bigcup_{i=1}^{\infty}E_i))$.

Corollary 1. *In order that $\mathfrak{R}(\mu)$ be μ-c. a., it is necessary and sufficient that $\mu(A)=\mu(A\frown B)+\mu(A\frown CB)$ for all A, $B\in\mathfrak{R}(\mu)$*[5].

4. Let $\{\mathfrak{M}_a\}$ be a family of classes of sets. By $(\mathfrak{M}_a)$ we denote *the smallest finitely additive class of sets containing all* $\mathfrak{M}_a$ and by $[\mathfrak{M}_a]$ *the smallest completely additive class of sets containing all* $\mathfrak{M}_a$.

Theorem 3. *If $\mathfrak{M}$ is μ-f. a., then $[\mathfrak{M}]$ is μ-c. a.*.

Proof. Let $\{K_n\}$ be a fundamental finite series. Then $(\{K_n\}, \mathfrak{M})=\mathfrak{M}_K$ is μ-f. a.[6]. Since μ is an outer measure we can easily prove that μ is a measure on $\mathfrak{M}_K$, i.e., if $E_i\in\mathfrak{M}_K$, $E_i\frown E_j=0$ $(i\neq j)$ and $\bigcup_{i=1}^{\infty}E_i\in\mathfrak{M}_K$, then $\mu(\bigcup_{i=1}^{\infty}E_i)=\sum_{i=1}^{\infty}\mu(E_i)$. For an arbitrary set $A\subset X$, we define $\mu^*(A)$ by $\mu^*(A)=\inf\{\sum_{i=1}^{\infty}\mu(E_i);\ E_i\in\mathfrak{M}_K,\ A\subset\bigcup_{i=1}^{\infty}E_i\}$. Then $[\mathfrak{M}_K]=[\{K_n\}, \mathfrak{M}]$ is μ^*-c. a., for every element of $[\mathfrak{M}_K]$ is μ^*-measurable in the sense of Carathéodory[7], and μ^* coincides with μ on $\mathfrak{M}_K$. It is clear that $\mu^*(A)\geqq\mu(A)$ for all A. We have only to show that $\mu^*(M)=\mu(M)$ for $M\in[\mathfrak{M}]$. Since $\mu(K_n)\leqq\mu(K_n\frown M)+\mu(K_n\frown CM)\leqq\mu^*(K_n\frown M)+\mu^*(K_n\frown CM)=\mu^*(K_n)=\mu(K_n)$, then we have $\mu^*(K_n\frown M)=\mu(K_n\frown M)$, and by Lemma 1 $\mu^*(M)=\mu(M)$.

Lemma 5. *Let $\mathfrak{M}$ and $\mathfrak{A}$ be μ-f. a. classes of sets such that $\mu_n(B)=\mu_n(B\frown M)+\mu_n(B\frown CM)$ (for all n) for all $B\in\mathfrak{A}$, $M\in\mathfrak{M}$ and for a fundamental finite series $\{K_n\}$. Then $(\mathfrak{M}, \mathfrak{A})$ is μ-f. a.*

Proof. Let E be an element of $(\mathfrak{M}, \mathfrak{A})$, then there exist a finite number of elements $B_i\in\mathfrak{A}$ and $M_i\in\mathfrak{M}$ $(i=1, 2, \ldots, k)$, such that $B_i\frown B_j=0$ $(i\neq j)$, $\bigcup_{i=1}^{k}B_i=X$ and $E=\bigcup_{i=1}^{k}(M_i\frown B_i)$. Then it holds:

$$(*)\qquad \mu_n(E)=\sum_{i=1}^{k}\mu_n(M_i\frown B_i)\,.$$

Because: Since $\mathfrak{M}\subset\mathfrak{R}(\mu)$, putting in Lemma 2 $A=K_n$, $E_i=B_i$ for $1\leqq i\leqq k$, $E_i=0$ for $i>k$ and $F=M$ $(M\in\mathfrak{M})$, we get:

$$(1)\qquad \mu_n(M)=\sum_{i=1}^{k}\mu_n(M\frown B_i)\qquad \text{for } M\in\mathfrak{M}\,.$$

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5) Cf. [E] Lemma 2.

6) Each element E of $\mathfrak{M}_K$ will be expressed in the form $E=(\bigcup_{i=1}^{n}(K_i-K_{i-1})\frown M_i)\smile(CK_n\frown M_{n+1})$ where $K_0=O$, $M_i\in\mathfrak{M}$; and $\mu(E)=\sum_{i=1}^{n}\mu((K_i-K_{i-1})\frown M_i)+\mu(CK_n\frown M_{n+1})$.

7) Cf. Halmos: Measure theory §12 Theorem A.

And also, putting $A=K_n\frown M$, $E_i=B_i$ for $1\leqq i\leqq k$, $E_i=0$ for $i>k$ and $F=M'$, where $M, M'\in\mathfrak{M}$, we get by (1)

$$(2)\qquad \mu_n(M\frown B_i)=\mu_n(M\frown M'\frown B_i)+\mu_n(M\frown CM'\frown B_i).$$

From (1) and (2) we have, putting $M=\bigvee_{j=1}^k M_j$, $\mu_n(\bigvee_{j=1}^k M_j)$ $=\sum_{i=1}^k\mu_n(\bigvee_{j=1}^k M_j\frown B_i)=\sum_{i=1}^k\{\mu_n(\bigvee_{j=1}^k M_j\frown B_i\frown M_i)+\mu_n(\bigvee_{j=1}^k M_j\frown B_i\frown CM_i)\}$ $=\sum_{i=1}^k\mu_n(B_i\frown M_i)+\sum_{i=1}^k\mu_n(\bigvee_{j\neq i}M_j\frown B_i)\geqq\mu_n(\bigvee_{i=1}^k(B_i\frown M_i))$ $+\mu_n(\bigvee_{i=1}^k\bigvee_{j\neq i}(M_j\frown B_i))\geqq\mu_n(\bigvee_{j=1}^k M_j\frown\bigvee_{i=1}^k B_i)=\mu_n(\bigvee_{j=1}^k M_j)$.
Therefore we obtain (*), where $E=\bigvee_{i=1}^k(M_i\frown B_i)$. Now let it be $E_\nu\in(\mathfrak{M},\mathfrak{A})$ $(\nu=1,2)$ and $E_1\frown E_2=0$. Then E_ν can be expressed in the form $E_\nu=\bigvee_{i=1}^k(M_i^\nu\frown B_i)$ $(\nu=1,2)$, where $B_i\in\mathfrak{A}$, $B_i\frown B_j=0$ $(i\neq j)$, $M_i^\nu\in\mathfrak{M}$ and $M_i^1\frown M_i^2=0$. Then by (*) and (2), $\mu_n(E_1\smile E_2)$ $=\mu_n(\bigvee_{i=1}^k((M_i^1\smile M_i^2)\frown B_i))=\sum_{i=1}^k\mu_n((M_i^1\smile M_i^2)\frown B_i)=\sum_{i=1}^k\{\mu_n(M_i^1\frown B_i)$ $+\mu_n(M_i^2\frown B_i)\}=\mu_n(E_1)+\mu_n(E_2)$.

Theorem 4. *Let $\mathfrak{M}$ and $\mathfrak{A}$ be μ-c. a. classes of sets. In order that $[\mathfrak{M},\mathfrak{A}]$ be μ-c. a., it is necessary and sufficient that $\mu_n(B)$ $=\mu_n(B\frown M)+\mu_n(B\frown CM)$ (for all n) holds for all $B\in\mathfrak{A}$, $M\in\mathfrak{M}$ and for a fundamental finite series $\{K_n\}$* [8].

Proof. The necessity of the condition is clear. The sufficiency follows from Theorem 3 and Lemma 5.

5. Let $\mathbf{M}=\{\mathfrak{M}_\alpha\}$ be *the system of all μ-c. a. classes of sets.* $\mathbf{M}$ will be an *ordered system* in such a way that "$\mathfrak{M}_\alpha\leqq\mathfrak{M}_\beta$ means that $\mathfrak{M}_\alpha\subset\mathfrak{M}_\beta$".

Theorem 5. *For each $\mathfrak{M}\in\mathbf{M}$ there exists a maximal element $\mathfrak{M}^*$ of $\mathbf{M}$ such that $\mathfrak{M}\leqq\mathfrak{M}^*$.*

Proof. Let $\{\mathfrak{M}_{\alpha(\lambda)}\}(\lambda\in\Lambda)$ be any linearly ordered sub-system of $\mathbf{M}$. Then $\{\mathfrak{M}_{\alpha(\lambda)}\}$ has an upper bound in $\mathbf{M}$. Because: One can easily prove that $(\bigvee_{\lambda\in\Lambda}\mathfrak{M}_{\alpha(\lambda)})=\bigvee_{\lambda\in\Lambda}\mathfrak{M}_{\alpha(\lambda)}$, and that $\bigvee_{\lambda\in\Lambda}\mathfrak{M}_{\alpha(\lambda)}$ is μ-f. a., hence by Theorem 3 $[\bigvee_{\lambda\in\Lambda}\mathfrak{M}_{\alpha(\lambda)}]$ is μ-f.a., which is an upper bound of $\{\mathfrak{M}_{\alpha(\lambda)}\}$. Thus, by Zorn's Lemma holds Theorem 5.

Theorem 6. *The union of all maximal μ-c. a. classes $\mathfrak{M}_\alpha^*\in\mathbf{M}$ coincides with $\mathfrak{N}(\mu)$: $\bigvee_\alpha\mathfrak{M}_\alpha^*=\mathfrak{N}(\mu)$* [9].

Proof. By Theorem 1: $\bigvee_\alpha\mathfrak{M}_\alpha^*\subset\mathfrak{N}(\mu)$. Let A be any element of

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8) It is clear that the intersection of an arbitrary number of μ-c. a. classes $\mathfrak{A}_\lambda(\lambda\in\Lambda)$ is also μ-c. a.. Theorem 4 can be extended as follows: "Let $\mathfrak{A}_\lambda$ $(\lambda\in\Lambda)$ be μ-c. a. classes of sets. Then the necessary and sufficient condition that $[\mathfrak{A}_\lambda(\lambda\in\Lambda)]$ be μ-c. a. is that for an arbitrary finite number of $\mathfrak{A}_{\lambda(i)}$ $(i=1,2,\ldots,k;\ k\geqq2)$ holds the relation: $\mu_n(\bigwedge_{i=1}^{k-1}A_i)=\mu_n(\bigwedge_{i=1}^{k-1}A_i\frown A_k)+\mu_n(\bigwedge_{i=1}^{k-1}A_i\frown CA_k)$, for all $A_i\in\mathfrak{A}_{\lambda(i)}$".

9) By Theorem 5, Theorem 6 and Theorem 7 holds: "the union of all classes $\mathfrak{M}_\alpha\in\mathbf{M}$ containing $\mathfrak{G}(\mu)$ coincides with $\mathfrak{N}(\mu)$". We can here replace $\mathfrak{G}(\mu)$ by $\mathfrak{C}(\mu)$ (by Theorem 1), Cf. [E] Lemma 3.

$\mathfrak{R}(\mu)$. Then (A, CA) is itself μ-c. a.. By Theorem 5 there exists a maximal $\mathfrak{M}_\alpha^*$ containing (A, CA). Therefore $\bigvee_\alpha \mathfrak{M}_\alpha^* \supset \mathfrak{R}(\mu)$.

Theorem 7. *The intersection of all maximal μ-c. a. classes $\mathfrak{M}_\alpha^* \in \mathbf{M}$ coincides with $\mathfrak{E}(\mu)$: $\bigwedge_\alpha \mathfrak{M}_\alpha^* = \mathfrak{E}(\mu)$.*

Proof: Since $\mathfrak{M}_\alpha^* \subset \mathfrak{R}(\mu)$, then by Theorem 4 $[\mathfrak{M}_\alpha^*, \mathfrak{E}(\mu)]$ is also μ-c. a.. Hence $\mathfrak{E}(\mu) \subset \mathfrak{M}_\alpha^*$. Therefore $\mathfrak{E}(\mu) \subset \bigwedge_\alpha \mathfrak{M}_\alpha^*$. Let us assume that $\mathfrak{E}(\mu) \neq \mathfrak{R}(\mu)$. Let A_1 be any element of $\mathfrak{R}(\mu) - \mathfrak{E}(\mu)$. Then there exists an $A_2 \in \mathfrak{R}(\mu)$ such that

$$(**) \qquad \mu(A_2) < \mu(A_2 \frown A_1) + \mu(A_2 \frown CA_1).$$

By Theorem 4 and Theorem 5 there exists a maximal element of $\mathbf{M}$, $\mathfrak{M}_0^*$, such that $\mathfrak{M}_0^* \supset [\mathfrak{E}(\mu), (A_2, CA_2)]$. But $A_1 \bar{\in} \mathfrak{M}_0^*$ by (**). Therefore $\bigwedge_\alpha \mathfrak{M}_\alpha^* \subset \mathfrak{E}(\mu)$. If $\mathfrak{E}(\mu) = \mathfrak{R}(\mu)$, we have also $\bigwedge_\alpha \mathfrak{M}_\alpha^* \subset \mathfrak{E}(\mu)$.

Corollary 2. *The necessary and sufficient condition that the ordered system* $\mathbf{M}$ *be a directed system, is that* $\mathfrak{E}(\mu) = \mathfrak{R}(\mu)$.

Proof. If there exists only one maximal element $\mathfrak{M}^*$ of $\mathbf{M}$, then by Theorem 6 and Theorem 7 holds: $\mathfrak{E}(\mu) = \mathfrak{M}^* = \mathfrak{R}(\mu)$. In this case $\mathbf{M}$ is a directed system. If there exist at least two different maximal elements $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ of $\mathbf{M}$, then $\mathbf{M}$ is not directed. By Theorem 6 and Theorem 7, we have in this case $\mathfrak{E}(\mu) \neq \mathfrak{R}(\mu)$.

6. **Remark.** *There are such cases that* $\mathfrak{C}(\mu) \neq \mathfrak{E}(\mu)$ *and* $\mathfrak{E}(\mu) \neq \mathfrak{R}(\mu)$, as the following examples show them[10].

Example 1. X consists of 3 points a, b, c; $X = (a, b, c)$, and μ is defined as follows: $\mu((a)) = 2$, $\mu((b)) = \mu((c)) = 3$, $\mu((b, c)) = \mu((c, a)) = \mu((a, b)) = 4$, $\mu(X) = 6$. Then $\mathfrak{E} = \mathfrak{R} = \{O, (a), (b), (c), X\}$, but $\mathfrak{C} = \{O, X\}$.

Example 2. $X = (a, b, c)$, μ is defined as follows: $\mu((a)) = \mu((b)) = \mu((c)) = 2$, $\mu((b, c)) = \mu((c, a)) = 3$, $\mu((a, b)) = 4$, $\mu(X) = 5$. Then $\mathfrak{R} = \{O, (a), (b), (b, c), (c, a), X\}$, but $\mathfrak{E} = \mathfrak{C} = \{O, X\}$.

10) Cf. [E] Remark 3.