5. On Some Problems of Birkhoff.

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In this paper we shall deal with problems 65 and 7 in Lattice Theory by G. Birkhoff[1] and discuss some axioms of lattices in connection with the latter problem.

§ 1. G. D. Birkhoff and G. Birkhoff[2] have developed a very brief set of postulates for distributive lattices.

Theorem. Any algebraic system which satisfies the following postulates for all a, b, c is a distributive lattice with I.

(1) $a \frown a = a$ for all a

$(2)_1$ $a \smile I = I$ $(2)_2$ $I \smile a = I$

$(3)_1$ $a \frown I = a$ $(3)_2$ $I \frown a = a$ for some I and all a

$(4)_1$ $a \frown (b \smile c) = (a \frown b) \smile (a \frown c)$ $(4)_2$ $(b \smile c) \frown a = (b \frown a) \smile (c \frown a)$.

Problem 65 is to prove or disprove the independence of the seven identities assumed as postulates in the above theorem.

We shall show that these identities are independent of each other.

I. *Independence of (1)*

We consider a system of three sets $I = \{1, 2, 3\}$, $a = \{1, 2\}$, $b = \{1\}$. About their join and meet operations, we take set-theoretical sum and intersection, except the case: $a \frown a = b$.

For this system we can easily show that the six postulates (2)–(4) are satisfied but (1) is not. Concerning $(4)_1$ we have, for instance

x	y	z	$x \frown (y \smile z)$	$(x \frown y) \smile (x \frown z)$
a	a	a	$a \frown (a \smile a) = a \frown a = b$,	$(a \frown a) \smile (a \frown a) = b \smile b = b$
a	a	b	$a \frown (a \smile b) = a \frown a = b$,	$(a \frown a) \smile (a \frown b) = b \smile b = b$.

All other cases are treated similarly. The other identities except (1) are verified easily.

II. *Independence of (2)*

Let $\mathfrak{S}$ be a family of all subsets of a set $\mathfrak{A}$ ($\mathfrak{S}$ does not contain null set.) Define join and meet operations as follows,

$a \smile b = a$

$a \frown b = ab$ (where ab means set-theoretical intersection of sets a, b)

1) G. Birkhoff: Lattice Theory, 1948.

2) G. D. Birkhoff and G. Birkhoff: Distributive postulates for systems like Boolean algebra. Trans. Am. Math. Soc. **60** (1946).

Then, it is obvious that the postulates (1), $(2)_2$, (3) are satisfied and moreover we have

$(4)_1$ $a \frown (b \smile c) = a \frown b, \quad (a \frown b) \smile (a \frown c) = a \frown b$

$(4)_2$ $(b \smile c) \frown a = b \frown a, \quad (b \frown a) \smile (c \frown a) = b \frown a,$

but, $(2)_1$ is not satisfied since $a \smile I = a \neq I$.

Similarly we can prove the independence of $(2)_2$ by defining join and meet operations as follows in the system constructed above, $a \smile b = b$, $a \frown b = ab$ (where the meaning of ab is the same as above.)

III. *Independence of (3).*

In the system $\mathfrak{S}$ defined in II, we modify the operations of join and meet as follows:

$a \smile b = a + b$ (set-theoretical sum of sets a, b)

$a \frown b = a$.

Then we can see the independence of $(3)_2$ by this system $\mathfrak{S}$. If in $\mathfrak{S}$ we make another modification: $a \smile b = a + b$, (set-theoretical sum of sets a, b). $a \frown b = b$, then we have a system which shows the independence of $(3)_1$.

IV. *Independence of* $(4)_1$

In a system of four elements a, b, c, I define join and meet as follows.

join: $I \smile x = x \smile I = I \quad (x = I,\ a,\ b,\ c)$

$x \smile y = y \smile x = I \quad (x \neq y;\ x,\ y = a,\ b,\ c)$

$x \smile x = x \quad (x = a,\ b,\ c)$

meet: $I \frown x = x \frown I = x \quad (x = I,\ a,\ b,\ c)$

$x \frown y = y \quad (x,\ y = a,\ b,\ c)$

For this system we can easily show that postulates (1) (2) (3) are satisfied. We can also prove the validity of $(4)_2$: $(y \smile z) \frown x = (y \smile x) \smile (z \frown x)$; for instance in case $x = y = a$, $z = b$ we have $(a \smile b) \frown a = I \frown a = a$, $(a \frown a) \smile (b \frown a) = a \smile a = a$, and other cases are treated similary. But $(4)_1$ is not satisfied; indeed we have $a \frown (b \smile c) = a \frown I = a$, $(a \frown b) \smile (a \frown c) = b \smile c = I$.

We can prove the independence of $(4)_2$ by defining $x \frown y = x$ instead of $x \frown y = y$ in the system constructed above.

§2. As is well known identities *L1*–*L4* completely characterize lattices.

L1	$x \frown x = x$	and	$x \smile x = x$
L2	$x \frown y = y \frown x$	and	$x \smile y = y \smile x$
L3	$x \frown (y \frown z) = (x \frown y) \frown z$	and	$x \smile (y \smile z) = (x \smile y) \smile z$
L4	$x \frown (x \smile y) = x$	and	$x \smile (x \frown y) = x$.

Problem 7 is to investigate the consequences of weakening $L1$ to $x\frown x = x\smile x$ and $L4$ to $x\frown(x\smile y) = x\smile(x\frown y)$. About this problem we obtain some results. Let L be a system satisfying

$\bar{L}1 \quad x\frown x = x\smile x = \bar{x}$

$L^*4 \quad x\frown(x\smile y) = x\smile(x\frown y) = x^*$ (independent of y)

and $L2$, $L3$. Let us put $\bar{L} = \{\bar{x};\ x\in L\}$, $L^* = \{x^*;\ x\in L\}$, $\bar{\bar{L}} = \{\bar{\bar{x}}; x\in L\}$. Then we obtain the following results.

I. *If $\bar{x} = x^*$ for every $x\in L$, then $\bar{L}$ is a lattice.*

In fact we have $\bar{x}\frown\bar{x} = \bar{x}\smile\bar{x} = x^* = \bar{x}$ since $\bar{x}\frown\bar{x} = (x\frown x)\frown(x\frown x)$ $= x\frown(x\frown(x\frown x)) = x\frown(x\smile(x\frown x)) = x^*$ by L^*4, and $\bar{x}\frown\bar{y} = \bar{y}\frown\bar{x}$ since $\bar{x}\frown\bar{y} = (x\frown x)\frown(y\frown y) = \overline{(x\frown y)} = \overline{(y\frown x)}$, and moreover $\bar{x}\frown(\bar{y}\frown\bar{z})$ $= \overline{x\frown(y\frown z)} = (\bar{x}\frown\bar{y})\frown\bar{z}$, and $\bar{x}\frown(\bar{x}\smile\bar{y}) = (\bar{x})^* = x^*\frown x^* = x^{**} = x^* = \bar{x}$ since $x^{**} = x^*\frown(x^*\smile x) = x^*\frown(x^*\frown x) = x\frown(x\frown A) = x^*$. Similarly we have the dual relations.

II. *L^* and $\bar{\bar{L}}$ are lattices for any L.*

Now we consider a system L satisfying $\bar{L}1$, $L2$, $L3$, L^*4 and the condition $\bar{x} = x^*$ for any x. Let $a, b, c\dots$ be distinct elements of this lattice $M = \bar{L}$. If we define $C_a = \{x;\ \bar{x} = a\}$, then C_a and C_b have no common elements, and we have $x\smile y\in C_{a\smile b}$, $x\frown y\in C_{a\frown b}$ for $x\in C_a$, $y\in C_b$. Furthermore if we assume the condition:

$(M) \qquad x\smile(x\smile y) = x\smile y, \qquad x\frown(x\frown y) = x\frown y$

then we have $\overline{x\smile y} = \overline{(x\smile y)\smile y} = \overline{x\smile\bar{y}} = \overline{x\smile(x\smile\bar{y})} = \bar{x}\smile\bar{y}$ and $\overline{x\frown y} = \bar{x}\frown\bar{y}$ by $L2$, $L3$ and (M). Hence we have $\overline{x\frown y} = a\frown b$, $\overline{x\smile y} = a\smile b$ for $x\in C_a$, $y\in C_b$ and we see that a system L satisfying $\bar{L}1$, $L2$, $L3$, L^*4 and (M) has the following structure. ((M) implies $\bar{x} = x^*$).

III. *Let M be a lattice. To each element a of M we correspond an abstract set C_a such that the intersection of C_a and M consists of only one element a, and C_a and C_b have no common elements for $a\neq b$. We define as follows:*

$$\overline{x\frown y} = a\frown b, \quad \overline{x\smile y} = a\smile b \quad \text{for} \quad x\in C_a,\ y\in C_b\ (a = b \text{ or } a\neq b).$$

Then the set-theoretical sum L of C_a, $a\in M$; $L = \sum_{a\in M} C_a$ satisfies $\bar{L}1$, $L2$, $L3$, L^*4 and (M).

Conversely any system satisfying these five conditions can be constructed as above.

The fact that (M) is not implied by $\bar{L}1$, $L2$, $L3$, L^*4 is shown by the following system of elements $\{a_i, b_j;\ i, j = 0, 1, 2\}$ in which join and meet of elements are defined as follows: $a\smile b = b$, $a\frown b = a$ and

1) In case $x\neq y$ and for $x, y = a, b$
   i) $x_i\asymp y_i = (x\asymp y)_{i-1}$, $i = 1, 2$ where $a_0 = a$, $b_0 = b$
   ii) $x_i\asymp y_j = (x\asymp y)$ for $i\neq j$; $i, j = 0, 1, 2$
2) $x_i\asymp x_j = x$ for $i \gtreqless j$ and $x = a, b$; $i, j = 0, 1, 2$.

In this system we can easily verify that $\bar{L}1$, $L2$, $L3$, L^*4 are satisfied, but (M) is not since $a_2 \cup (a_2 \cup b_2) = a_2 \cup b_1 = b \neq a_2 \cup b_2 = b_1$.

Remark. (1), $\bar{L}1$, $L2$, $L3$, (M) and (γ) imply L^*4, where (γ): $x \cup y = y \cup y$ implies $x \cap y = x \cap x$ and conversely. In fact we have $x \cup y = x \cup (x \cup y) = (x \cup y) \cup (x \cup y)$ by (M), $L3$, and $x \cap (x \cup y) = x \cap x$ by (γ). Similarly $x \cup (x \cap y) = x \cup x$. Hence we have

$x \cap (x \cup y) = x \cup (x \cap y)$ by $\bar{L}1$.

(2) If we take a closure operation $A \rightarrow \tilde{A}$ for the subsets A of an abstract set R and define,

$A \cup B = \tilde{A} + \tilde{B}$ (set-theoretical sum)

$A \cap B = \tilde{A} \cdot \tilde{B}$ (set-theoretical intersection),

then we have a concrete example satisfying $\bar{L}1$, $L2$, $L3$, L^*4 and (M).

(3) Any system satisfying the condition (α) besides $\bar{L}1$, $L2$, $L3$, L^*4, (M) is a lattice, where (α): $\bar{x} = \bar{y}$ implies $x = y$. Indeed we have $\bar{x} = \overline{x \cup x}$, $\bar{x} = \bar{x} \cap (\bar{x} \cup \bar{y}) = \overline{x \cap (x \cup y)}$. since $\bar{L}$ is a lattice, hence we have $x = x \cup x$, $x = x \cap (x \cup y)$. Similarly we have $x \cup (x \cap y) = x$.

Hitherto we have assumed that x^* is independent of y, we now treat the following postulates

$L'4$: $x \cap (x \cup y) = x \cup (x \cap y)$

which is weakening of L^*4.

IV. *Any system satisfying L1, L2, L3, L′4 is not always a lattice.*

We shall show this fact by the following example.

Let $L = \{a, b, c\}$, $a < b < c$, and define join and meet as usual, except the case: $a \cup b = a \cap b = b$. In this system we can see that $L1$, $L2$, $L3$, $L'4$ are satisfied. However $L4$ does not hold since $a \cap (a \cup b) = a \cap b = b \neq a$.

V. *Any system satisfying* $\bar{L}1$, $L2$, $L3$, $L'4$, *and the condition* (α) *is a lattice, where*

(α): $x \cap y = x \cup y$ *implies* $x = y$.

Proof. By $\bar{L}1$, $L'4$ we get $x \cup \bar{x} = x \cup (x \cap x) = x \cap (x \cup x) = x \cap \bar{x}$. Hence $x = \bar{x}$ by (α).

Since $x \cup (x \cup (x \cap y)) = x \cup (x \cap y) = x \cap (x \cup y) = x \cap (x \cap (x \cup y))$ by $L'4$, $x = \bar{x}$, we have $x \cup (x \cap y) = x$ by (α).

§3. We shall now find some conditions instead of $L4$ for a system to be a lattice besides $L2$, $L3$.

I. *A system satisfying any one of the following class of conditions besides L2, L3 is a lattice,*

(1) $(\bar{L}1, (M)_1, (\gamma)^*)$

(2) $(L'4, (M)_1, (\gamma)_1^*)$

(3) $(L'4, (M)_2, (\gamma)_2^*, x \cup x = x)$

(4) $(L'4, (M)_2, (\beta)_1)$

where

$(M)_1$: $x \cup (x \cup y) = x \cup y$
$(M)_2$: $x \cap (x \cap y) = x \cap y$
$(\gamma)^*$: $x \cup y = y \cup y$ implies $x \cap y = x$ and conversely
$(\gamma)^*_1$: $x \cup y = y \cup y$ implies $x \cap y = x$
$(\gamma)^*_2$: $x \cap y = x$ implies $x \cup y = y \cup y$
$(\beta)_1$: $x \cap y = x$ implies $x \cup y = y$.

Proof. (1) By $L3$, $(M)_1$, we have $x \cup (x \cup y) = (x \cup y) \cup (x \cup y)$, and hence $x \cap (x \cup y) = x$ (i) by $(\gamma)^*_1$.
From $x \cup x = x \cup x$, we get $x \cap x = x$ (ii) by $(\gamma)^*_1$.
By $L3$, (ii), $x \cap y = x \cap (x \cap y)$,
then $x \cup (x \cap y) = x \cup x$ (iii) by $(\gamma)^*_2$, $L2$.
Hence we have $x \cup (x \cap y) = x$ by (ii), (iii), $\bar{L}1$.

(3) Since $x \cap (x \cap y) = x \cap y$ by $(M)_2$, then by $(\gamma)^*_2$, $L2$, $(x \cap y) \cup x = x \cup x$. By $L'4$ $x \cap (x \cup y) = x \cup x$, and we have $L4$ from the condition $x \cup x = x$. The proof for (2) and (4) are omitted.

Remark. $L'4$, $(M)_1$, $(\gamma)^*_1$ besides $L2$, $L3$ imply $(\gamma)^*_2$, $(M)_2$. Indeed, if $x \cap y = x$ then we have $y \cup x = y \cup (x \cap y) = y \cup (y \cap x)$ by $L2$. On the other hand $y \cup (y \cap x) = y \cup y$, for $x \cap (x \cup y) = x$ by $(M)_1$, $L3$, $(\gamma)^*_1$, then $x \cup (x \cap (x \cup y)) = x \cup x$. By $L'4$, $(M)_1$ $x \cup (x \cap (x \cup y)) = x \cup (x \cup (x \cap y)) = x \cup (x \cap y)$, hence we have $x \cup (x \cap y) = x \cup x$. Accordingly we have $y \cup (y \cap x) = y \cup y$. Hence we get $(\gamma)^*_2$. $(M)_2$ is trivial. However $(\gamma)^*_1$, $(M)_1$ will not be implied by $L'4$, $(M)_2$, $(\gamma)^*_2$, $L2$, $L3$.

II. *The four identities $L2$, $L3$, (M), (β) characterize a lattice, where*

(β): $x \cup y = y$ implies $x \cap y = x$ and conversely,

(M): $x \cup (x \cup y) = x \cup y$, $x \cap (x \cap y) = x \cap y$.

Proof. By (M), $x \cup (x \cup y) = x \cup y$
hence we get $x \cap (x \cup y) = x$ by (β).
Similarly we have $x \cup (x \cap y) = x$ by (β), (M).
Now we shall prove the independence of these identities.

(1) We shall show the interesting example for the indpendence of (M). Let $\gamma = \{0, 2, 3, \cdots\}$ and define join and meet as follows.

$a \cup b = a + b$, $a \cap c = ab$ (arithmetical sum and product)

In this system $L2$, $L3$ is trivial. It is easily shown that (M) is not satisfied. Concerning (β), we have

$a + b = b \rightleftarrows a = 0 \rightleftarrows ab = a$ since $b \neq 1$.

(2) We can see that the independence of (β) is obtained by the same example as in IV of §2.

(3) Independence of $L2$.

We consider two isomorphic lattices K, K', and define join and meet of elements between K and K' as follows.

$$a \asymp b' = a \asymp b, \quad b' \asymp a = (b \asymp a)' \quad \text{for} \quad a, b \in K;\ a', b' \in K',$$

where the elements $a, b, \cdots$ of K corresponds to $a', b', \cdots$ of K'. In this system $L2$ does not hold, since $a \cup b' = a \cup b$, $b' \cup a = (b \cup a)' = (a \cup b)'$. However $L3$, (M) hold. The validity of (β) is evident since the relation $x \cup y = y$ holds only for $x, y \in K$ or $x, y \in K'$.

(4) Independence of $L3$

Let $L = \{a, b, c, I\}$, $a<b<c<I$ and define join and meet as usual, except the following case: $a \cup b = b \cup a = I$. Then we can prove the validity of $L2$, (M), (β) in this system. However $L3$ is not satisfied since $a \cup (b \cup c) = a \cup c = c$, $(a \cup b) \cup c = I \cup c = I$.