

25. *Probability-theoretic Investigations on Inheritance.* VII₁. *Non-Paternity Problems.*¹⁾

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1. Preliminaries and notations.

In conformity with a principle of Mendelian inheritance, any gene contained in neither of parents' types can never appear in a type of their child. That is to say, any gene appearing in a type of child must necessarily be contained in at least one type of its parents. Moreover, any gene contained in a type of a parent can appear in a type of child, and the passage of each gene takes place equally probably. Various results established have always been based upon this assumption. For instance, the table listed in § 3 of I represents such circumstances very well.

In view of the principle, if a child has a gene not contained in type of its mother, then the gene must surely originate from its father, so that it then becomes possible to conclude the non-paternity of a putative father as being not a true father. This is the reason why inheritance phenomena, especially those of human blood types the inheritance modes of which have been quite clarified, can be and have really been applied to establish non-paternity, from medico-legal standpoint, in cases of bastardization.

It is of much practical importance whether recessive genes are existent or not. If there exists a recessive gene, an individual representing a dominant character can be, besides of homozygote, also of heterozygote. Hence, for instance, in case of pair of a mother and her child having the same dominant character in common, if their types are both known to be of homozygote, a man having the corresponding recessive character would be proved to be not a true father. But, such a decision is possible only upon genotypes but impossible upon only phenotypes, since there exists another possibility of the same triple on phenotypes; namely, the possibility of

1) Y. Komatu, Probability-theoretic investigations on inheritance. I. Distribution of genes; II. Cross-breeding phenomena; III. Further discussions on cross-breeding; IV. Mother-child combinations; V. Brethren-combinations; VI. Rate of danger in random blood transfusion. *Proc. Jap. Acad.* **27** (1951), I: 371-377; II: 378-383, 384-387; III: 459-465, 466-471, 472-477, 478-483; IV: 587-592, 593-597, 598-603, 605-610, 611-614, 615-620; V; **28** (1952), VI: 54-58. These will be referred to as I, II, III, IV, V, VI.

mother and child being both heterozygotic. Consequently, with the aid of only phenotypes which alone we can know directly by practical observations, certain cases favorable for non-paternity proof upon genotypes must be dismissed undecidedly. On the contrary, if there is no recessive gene, such a circumstance never takes place. Every gene being then equivalent each other in intensity, every phenotype consists of a unique genotype. Consequently, given a mother-child combination, if non-paternity proof is possible upon genotypes, so is also upon phenotypes. On the other words, it is impossible upon genotypes, provided so is upon phenotypes.

In the present chapter we shall discuss the problem stated as follows: *Given a mother-child combination, how much is the probability in which we can prove non-paternity of a putative father?*

We now explain notations which will be used later. Let

$$(1.1) \quad \{p_i\} \quad (i=1, \dots, m)$$

be a set of probabilities of m mutually exclusive events exhausting all the possible cases. We introduce the *power-sum*, i. e., the sum of power with homogeneous degree ν :

$$(1.2) \quad S_\nu = \sum_{i=1}^m p_i^\nu \quad (\nu=0, 1, 2, \dots);$$

in particular, by taking the assumption into account,

$$(1.3) \quad S_0 = m, \quad S_1 = 1,$$

and S_ν decreases as ν increases. It is well known that any symmetric polynomial with respect to (1.1) is expressible also in terms of (1.2) alone as a polynomial. For example, (cf. below)

$$(1.4) \quad \sum_{\substack{i, j=1 \\ i \neq j}}^m p_i p_j = S_1^2 - S_2 (=1 - S_2).$$

Let further, in general, $\{f_{ij}\}$ ($i, j=1, \dots, m$) be a set of m^2 quantities not necessarily symmetric with respect to both suffices i and j . We then introduce a convention concerning summation defined as

$$(1.5) \quad \sum'_{i, j} f_{ij} = \sum_{\substack{i, j=1 \\ i < j}}^m f_{ij}.$$

Making use of this convention, we may write, for instance,

$$(1.6) \quad \sum_{\substack{i, j=1 \\ i \leq j}}^m f_{ij} = \sum_{i=1}^m f_{ii} + \sum'_{i, j} f_{ij}.$$

If, in particular, the set $\{f_{ij}\}$ satisfies the *symmetry relation* $f_{ij} = f_{ji}$ for any pair of suffices, it is evident that the relation

$$(1.7) \quad \sum'_{i, j} f_{ij} = \frac{1}{2} \left(\sum_{i, j=1}^m f_{ij} - \sum_{i=1}^m f_{ii} \right)$$

holds good²⁾. For instance, we get

$$\sum_{\substack{i,j=1 \\ i \neq j}}^m p_i p_j = 2 \sum'_{i,j} p_i p_j = \sum_{i,j=1}^m p_i p_j - \sum_{i=1}^m p_i^2 = S_1^2 - S_2$$

in conformity with (1.4). Regardless whether $\{f_{ij}\}$ is or is not symmetric with respect to suffices, the set $\{f_{ij} + f_{ji}\}$ being always symmetric, we get, in view of (1.7), an identical relation

$$(1.8) \quad \sum'_{i,j} (f_{ij} + f_{ji}) = \sum_{i,j=1}^m f_{ij} - \sum_{i=1}^m f_{ii}.$$

Besides (1.1) let further

$$(1.9) \quad \{p'_i\} \quad (i=1, \dots, m)$$

be a set of similar nature. We make, in general, an agreement that any expression, obtained from an expression F concerning (1.1) after replacing all the p_i 's by the corresponding p'_i 's, will be denoted by $[F]'$; namely, we shall put

$$(1.10) \quad [F]' = [F]^{(p')=(p')}.$$

For instance, the power-sum of (1.9) may be written in the form

$$(1.11) \quad S'_\nu \equiv \sum_{i=1}^m p_i'^\nu = [S_\nu]' \quad (\nu=0, 1, 2, \dots).$$

We further generalize the notation of both power-sums (1.2) and (1.11) to that of mutual power-sum

$$(1.12) \quad S_{\mu,\nu} \equiv \sum_{i=1}^m p_i^\mu p_i'^\nu \quad (\mu, \nu=0, 1, 2, \dots);$$

so to speak, the scalar product of power-vectors $\{p_i^\mu\}$ and $\{p_i'^\nu\}$. We see evidently

$$(1.13) \quad S_{\mu,0} = S_\mu, \quad S_{0,\nu} = S'_\nu \equiv [S_\nu]'$$

$$(1.14) \quad [S_{\mu,\nu}]^{(p')=(p)} = S_{\mu+\nu}.$$

By the way, we introduce, besides (1.5), similar conventions defined by

$$(1.15) \quad \sum'_{\substack{h,k \neq i \\ h, k \neq i; h < k}} f_{hk} = \sum_{\substack{h,k=1 \\ h < k}}^m f_{hk}, \quad \sum'_{\substack{h,k \neq i, j \\ h, k \neq i, j; h < k}} f_{hk} = \sum_{\substack{h,k=1 \\ h < k}}^m f_{hk} \quad (i \neq j),$$

which will be used in a later chapter. If, in particular, the symmetric relation $f_{hk} = f_{kh}$ is satisfied for any pair of suffices, then it is easy to see that the relations analogous to (1.7) hold good:

$$(1.16) \quad \begin{aligned} \sum'_{h, k \neq i} f_{hk} &= \frac{1}{2} \left(\sum_{h, k=1}^m f_{hk} - \sum_{h=1}^m (2f_{ih} + f_{hh}) \right) + f_{ii}, \\ \sum'_{h, k \neq i, j} f_{hk} &= \frac{1}{2} \left(\sum_{h, k=1}^m f_{hk} - \sum_{h=1}^m (2f_{ih} + 2f_{jh} + f_{hh}) \right) + f_{ii} + f_{jj} + f_{ij} \quad (i \neq j). \end{aligned}$$

—To be continued—

2) Cf. also a remark which will be stated at a later place immediately subsequent to (2. 19) or to (3. 6).