17. Note on Dirichlet Series (IX). Remarks on J. J. Gergen-S. Mandelbrojt's Theorems.

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(1) **Introduction.** Let us put

$$(1.1)\quad F(s)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \qquad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2\ldots<\lambda_n\to\infty).$$

If (1.1) is simply convergent in the whole plane, then (1.1) defines an integral function. Now we shall begin with

Definition. *Let (1.1) be simply convergent in the whole plane. Suppose that (1.1) assumes every value, except perhaps two (∞ included), infinitely many times, in any angular domain: $|\arg(s-s_0)-\theta|<\varepsilon$, where s_0: fixed point, ε: any positive number. Then* $\arg(s-s_0)=\theta$ *is called Julia's direction with respect to s_0. For brevity, we denote it by $J(s_0:\theta)$-direction.*

In the last part of their interesting note ([1] theorems V-VII), J. J. Gergen-S. Mandelbrojt established the existence of $J(0:\theta)$-directions under some assumptions. In this note, we shall prove the existence of $J(s_0:\theta)$-directions under hypotheses somewhat different from their ones.

(2) **Theorem I.** In this section, we shall prove

Theorem I. Let (1.1) be simply convergent in the whole plane, and not be a constant. Then, for any given point $s_0=\sigma_0+it_0$, there exist two $J(s_0:\pm\pi/2)$-directions, provided that (1.1) is uniformly convergent for $\sigma_0-\alpha\leqq\sigma$, where α: sufficiently small positive constant.

From this theorem immediately follows

Corollary. Let (1.1) be uniformly convergent in the whole plane, and not be a constant. Then, for any given point s_0, there exist two $J(s_0:\pm\pi/2)$-directions.

Formerly the author proved this corollary under the absolute convergence in the whole plane, but recently Prof. A. Wintner kindly remarked to him that this corollary is valid.

In order to establish theorem 1, we need some lemmas.

Lemma I. (H. Bohr, [2] p. 49) *If (1.1) is uniformly convergent for $\sigma_0\leqq\sigma$, then to any bounded domain Δ interior to this half-plane, and to any given $\varepsilon(>0)$, corresponds a sequence of numbers $\{\tau_p\}$ such that, for any s contained in Δ we have*

$$|F(s+i\tau_p)-F(s)|<\varepsilon \qquad (p=\pm1,\ \pm2,\ \ldots),$$

where

$$\varliminf_{p=\pm\infty}(\tau_{p+1}-\tau_p)>0\,, \qquad \varlimsup_{p=\pm\infty}|\tau_p|/p<+\infty\,.$$

Lemma II. Under the hypotheses of Theorem 1, for any given ε (>0), *(1.1) is unbounded in* $\pi/2\leqq|\arg(s-s_0)|\leqq\pi/2+\varepsilon$.

Proof. We shall show that, in $\pi/2\leqq\arg(s-s_0)\leqq\pi/2+\varepsilon$, (1.1) is unbounded. The unboundedness in $-\pi/2\geqq\arg(s-s_0)\geqq-(\pi/2+\varepsilon)$ is similarly established. It is sufficient to show that the boundedness in $\pi/2\leqq\arg(s-s_0)\leqq\pi/2+\varepsilon$ ascertains the boundedness in the whole plane, because by Liouville's well-known theorem, (1.1) reduces to a constant, which contradicts the hypotheses. For its purpose, it further suffices to prove the uniform boundedness in the circle C: $|s-s_0|\leqq R$, where R is an arbitrary large number.

Since $F(s)$ is evidently bounded for $\sigma_0-a\leqq\sigma$, there exists a constant K such that

$$(2.1)\qquad |F(s)|<K \quad \text{for} \quad -\pi/2\leqq\arg(s-s_0)\leqq\pi/2+\varepsilon\,.$$

Putting $\Delta:|s-s_0|\leqq r<a$, $\varepsilon=\varepsilon_n\,(\varepsilon_1>\varepsilon_2>\ldots>\varepsilon_n\to0)$, in Lemma 1, we can conclude that there exists a sequence $\{\tau_{p_n}\}$ such that

(2.2) $|F(s+i\tau_{p_n})-F(s)|<\varepsilon_n$ for s contained in Δ, where $\tau_{p_1}<\tau_{p_2}<\ldots<\tau_{p_n}<\ldots\to+\infty$. Setting $F_n(s)=F(s+i\tau_{p_n})$, by (2.2) $F_n(s)$ tends uniformly to $F(s)$ in Δ. On the other hand, for a sufficiently large $n>N(R)$, we have easily $|F_n(s)|<K$ in C. Hence, by Vitali's well-known theorem $F_n(s)$ also tends uniformly to $F(s)$ in C, so that we have evidently $|F(s)|<K$ in C, which is to be proved.

Proof of Theorem I. By Lemma 2, for any given $\varepsilon(>0)$, $F(s)$ is unbounded in $D(\varepsilon)$: $\pi/2\leqq\arg(s-s_0)\leqq\pi/2+\varepsilon$. Hence there exists evidently a sequence $\{S_n\}$ $(n=1,2,\ldots)$ such that

$$(2.3)\qquad \begin{cases}(\text{i}) & S_n\in D(\varepsilon),\quad |S_1|<|S_2|<|S_3|<\ldots<|S_n|\to+\infty\\ (\text{ii}) & |F(S_n)|\to+\infty \quad \text{as} \quad n\to+\infty\,.\end{cases}$$

Now let us consider the function-family $F_k(s)=F(s_0+2^k(s-s_0))$ in

$$D:\ 1/2\leqq|s-s_0|\leqq1,\quad |\arg(s-s_0)-\pi/2|\leqq\varepsilon\,.$$

By (2.3) we can easily find two sequences $\{k_n\}$ (integers), $\{s_n\}$ such that

$$(2.4)\qquad \begin{cases}(\text{i}) & S_n=s_0+2^{k_n}(s_n-s_0)\,,\ s_n\in D,\\ (\text{ii}) & |F_{k_n}(s_n)|=|F(S_n)|\to+\infty \ \text{as}\ n\to+\infty\,.\end{cases}$$

On the other hand, on account of the uniform convergence of $F(s)$ in $\sigma_0-a\leqq\sigma$, there exists a constant K such that

$$(2.5)\qquad |F_k(s)|<K \quad \text{for} \quad s\in D,\ \pi/2\geqq\arg(s-s_0)\geqq\pi/2-\varepsilon\,.$$

Then, the function-family $\{F_k(s)\}$ is not normal in

$$D':\ 1/2-\delta<|s-s_0|<1+\delta\ (\delta>0),\ |\arg(s-s_0)-\pi/2|<2\varepsilon\,.$$

In fact, by (2.4), (2.5), any partial sequence of $\{F_{k_n}(s)\}$ neither tends uniformly to ∞ in D', nor tends uniformly to a finite analytic function, so that $\{F_k(s)\}$ is not normal in D'. Hence there exists at least one not-normal point in D'. Thus, in $|\arg(s-s_0)-\pi/2|<2\varepsilon$, $F(s)$ assumes every value, except perhaps two (∞ included), infinitely many times. Since ε is arbitrary, $\arg(s-s_0)=\pi/2$ is $J(s_0:+\pi/2)$-direction. By the similar arguments $\arg(s-s_0)=-\pi/2$ is $J(s_0:-\pi/2)$-direction.

(3) **Theorem II.** Here we shall generalize Theorem 1 as follows:

Theorem II. Let (1.1) be simply convergent in the whole plane, and not be a constant. Then, for any given point $s_0=\sigma_0+it_0$, there exist two $J(s_0: \pm\pi/2)$-directions, provided that (1.1) is uniformly bounded for $\sigma_0-\alpha\leqq\sigma$, where α: sufficiently small positive constant.

Theorem 1 immediately follows from Theorem 2. For its proof, we need

Lemma III. In Lemma 1, the uniform convergence of (1.1) for $\sigma_0\leqq\sigma$ can be replaced by the uniform boundedness of (1.1) for $\sigma_0\leqq\sigma$.

Proof. Since Δ is interior to the half-plane $\sigma_0\leqq\sigma$, we can choose a sufficiently small positive constant ε' such that Δ is contained in $\sigma_0+\varepsilon'\leqq\sigma$. Then, by the well-known theorem ([2] p. 11, XI), we have

$$F(s)=\sum_{\lambda_n<u}a_n\exp(-\lambda_n s)\ (1-\exp(\lambda_n-u))^k+0(\exp\ (-\delta u)) \tag{3.1}$$

uniformly with respect to s contained in Δ, where k: positive integer, $\sigma_0+\varepsilon'\leqq\sigma$, $\delta>0$. Hence, we get

$$F(s)=\lim_{u\to+\infty}\sum_{\lambda_n<u}a_n\exp(-\lambda_n s)\ (1-\exp(\lambda_n-u))^k \tag{3.2}$$

uniformly with respect to s contained in Δ. On the other hand, $\sum_{\lambda_n<u}a_n\exp(-\lambda_n s)\ (1-\exp(\lambda_n-u))^k$ is an analytic and almost periodic function of s, so that by (3.2) and H. Bohr's theorems, $F(s)$ is also an analytic and almost periodic function of s in Δ. Hence, from the almost-periodicity of $F(s)$ in Δ, we can conclude the existence of the sequence of $\{\tau_p\}$ satisfying the same properties as Lemma 1.

q.e.d.

On account of Lemma 3, under the hypotheses of Theorem 2, Lemma 2 is also valid. Hence, by the entirely similar arguments as Theorem 1, Theorem 2 can be established.

(4) **Remark.** If we assume only the simple convergence of (1.1) in the whole plane, then what we can say about the existence of Julia's directions? Concerning this problem, we can prove

Theorem III. Let (1.1) be simply convergent in the whole plane, and not be a constant. Then, for any given point s_0, there exists at least one $J(s_0: \theta)$-direction with $|\theta-\pi|\leqq\pi/2$.

Its proof is trivial: Since (1.1) represents an integral function, there exists at least one Julia's direction. On the other hand, (1.1) is uniformly convergent in $|\arg(s-s_0)| \leqq \alpha < \pi/2$ ([2] p. 2), so that (1.1) is evidently uniformly bounded in this angular domain. Hence, Julia's direction can not exist in $|\arg(s-s_0)| < \pi/2$, which is to be proved.

References.

1) J. J. Gergen-S. Mandelbrojt: "On entire functions defined by a Dirichlet series." Amer. Jour. Math. Vol. **53** (1931).

2) G. Valiron: "Théorie générale des séries de Dirichlet." Mémorial des sciences mathématiques, Fasc. **17** (1926).