40. Probability-theoretic Investigations on Inheritance. VIII$_1$. Further Discussions on Non-Paternity Problems.

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1. Problems to be discussed.

In the last chapter of preceding Note[1], we have discussed various problems on proving non-paternity, with the aid of probabilities on mother-child combinations with respect to one child family. The problems treated there have concerned, however, exclusively those in which the paternity for a child is deniable by a third person against its parents or its mother. More precisely spoken, a typical problem has been to determine at how many rate a person can assert his non-paternity based upon an inheritance character under consideration, if he falls under suspicion to be a father of a child produced from a couple.

Besides the problems of this sort, there may occur those of another sort, which will be discussed in the present chapter; namely, *non-paternity problems amongst a couple*. To speak more precisely, a typical problem is as follows: If a wife has become intimate with a man and given birth to a child, at how many rate can her husband assert his non-paternity, based upon an inherited character, against the child? Hence, while the previous problem has concerned the non-paternity of a *defendant* in case of adultery, the present problems concerns that of a *plaintiff*.

From a view-point of the whole probability of proving non-paternity, both problems lead, of course, to quite an identical result. Indeed, in either of the problems, given a pair of a woman and her child, it is to be determined, at how many rate a man being not a father of the child—a third man in the previous problem or a husband of the woman in the present problem—can be proved as really not to be a true father. Consequently, every sub-pro-

1) Y. Komatu, Probability-theoretic investigations on inheritance. I. Distribution of genes; II. Cross-breeding phenomena; III. Further discussions on cross-breeding; IV. Mother-child combinations; V. Brethren-combinations; VI. Rate of danger in random blood transfusion; VII. Non-paternity problems. Proc. Jap. Acad. **27** (1951), I. 371-377; II. 378-383, 384-387; III. 459-464, 466-471, 472-477, 478-483; IV. 587-592, 593-597, 598-603, 605-610, 611-614, 615-620; V.; **28** (1952), VI. 54-58; VII. 102-104, 105-108, 109-111, 112-115, 116-120, 121-125. These will be referred to as I; II; III; IV; V; VI; VII.

bability with respect to the given type of a woman in question coincides also each other. The results which will be afresh obtained by discussions of the present problem are thus the sub-probabilities with respect to pairs of matings.

By summing up the sub-probabilities under consideration, we shall again confirm a result on the whole probability derived in the preceding chapter. Besides a decomposition of the whole probability into such sub-probabilities, we shall consider later a decomposition with respect to type of child, by means of which a mutual relation between two decompositions will be made still more clear. In fact, the position of child will show a strong similarity in both problems.

We now consider, as before, an inherited character consisting of m allelomorphic genes $A_i (i=1, \ldots, m)$. Given a fixed mating, the number of possible types of a child is then evidently equal to 1 or 2 if mother is homozygotic and to 2 or 3 or 4 if she is heterozygotic. On the other hand, as shows a table on mother-child combinations listed in § 1 of IV, the number of possible types of a child produced from a fixed mother of homozygotic or heterozygotic type is equal to m or $2m-1$, respectively. Hence, the respective differences $m-1$ or $m-2$ and $2m-3$ or $2m-4$ or $2m-5$ represent the numbers of possible types of a child against whom the husband can assert non-paternity, according to the wife (the mother of child) of homozygotic or heterozygotic type.

As stated in (1.1) of I, there exists, in general, $\frac{1}{2}m(m+1)$ possible genotypes. However, those except the above-stated m or $2m-1$ genotypes of child are out of question. Since those exceptional types can never appear in a child of a given mother, the protest against her unchastity is then quite unreasonable so that she must be released from responsibility concerning unchastity.

2. Sub-probability with respect to a type of wife.

If a wife and her husband are both of the same homozygote, A_{ii} say, then a child produced by this couple must be always also of the same type. On the other hand, possible types of child produced by a mother A_{ii} are, in general, those containing the gene A_i, i.e., A_{ii} and $A_{ij} (j \neq i)$. Hence, the husband can assert his non-paternity against any heterozygotic child $A_{ij} (j \neq i)$ among them. The probability in which a mother A_{ii} produces a child A_{ij} is equal to p_j. In fact, while in the table in § 1 of IV the probability $\pi(ii; ij)=p_i^2 p_j$, the frequency $\bar{A}_{ii}=p_i^2$ of a wife (mother of child) being also taken into account, has been listed, a fixed type of wife is considered in the present problem and hence the value $\pi(ii; ij)/\bar{A}_{ii} =p_j$ must be used; cf. (1.27) of IV.

Now, given a couple of wife $A_{ij}(i \leqq j)$ and her husband $A_{hk}(h \leqq k)$, let the probability in which the husband can assert his non-paternity against a child produced by the wife together with a man chosen at random with respect to types be denoted by

$$(2.1) \qquad U(ij, hk) \qquad (i, j, h, k=1, \ldots, m;\ i \leqq j;\ h \leqq k);$$

the symmetry relations analogous to (1.3) of IV being taken into account. Then, the above argument leads to

$$(2.2) \qquad U(ii, ii) = \sum_{j \neq i} \pi(ii; ij)/\bar{A}_{ii} = \sum_{j \neq i} p_j = 1 - p_i.$$

If a couple consists of a wife A_{ii} and her husband $A_{ih}(h \neq i)$, then possible types of a child produced by this couple are A_{ii} and A_{ih}, and hence we obtain

$$(2.3) \qquad U(ii, ih) = \sum_{j \neq i, h} p_j = 1 - p_i - p_h \qquad (h \neq i);$$

we get similarly

$$(2.4) \qquad U(ii, hh) = p_i + \sum_{j \neq i, h} p_j = 1 - p_h \qquad (h \neq i),$$

$$(2.5) \qquad U(ii, hk) = p_i + \sum_{j \neq i, h, k} p_j = 1 - p_h - p_k \qquad (h, k \neq i;\ h \neq k).$$

The cases of heterozygotic wives can be treated also in a similar manner. If a couple consists of a wife $A_{ij}(i \neq j)$ and her husband A_{ii}, then he can assert his non-paternity against any child of types, produced by her, except A_{ii} and A_{ij}, i.e., against $A_{jj}, A_{ik}, A_{jk}(k \neq i, j)$. Hence, we get

$$(2.6) \qquad \begin{aligned} U(ij, ii) &= \Big(\pi(ij; jj) + \sum_{k \neq i, j} (\pi(ij; ik) + \pi(ij; jk))\Big)\Big/\bar{A}_{ij} \\ &= \tfrac{1}{2}p_j + \sum_{k \neq i, j} (\tfrac{1}{2}p_k + \tfrac{1}{2}p_k) = 1 - p_i - \tfrac{1}{2}p_j \qquad (i \neq j). \end{aligned}$$

In quite a similar manner, we obtain the following results:

$$(2.7) \qquad U(ij, ij) = \sum_{k \neq i, j} (\tfrac{1}{2}p_k + \tfrac{1}{2}p_k) = 1 - p_i - p_j \qquad (i \neq j),$$

$$(2.8) \qquad U(ij, ih) = \tfrac{1}{2}p_j + \sum_{k \neq i, j, h} (\tfrac{1}{2}p_k + \tfrac{1}{2}p_k) = 1 - p_i - \tfrac{1}{2}p_j - p_h \quad (i \neq j;\ h \neq i, j),$$

$$(2.9) \qquad \begin{aligned} U(ij, hh) &= \tfrac{1}{2}p_i + \tfrac{1}{2}p_j + \tfrac{1}{2}(p_i + p_j) + \sum_{k \neq i, j, h} (\tfrac{1}{2}p_k + \tfrac{1}{2}p_k) = 1 - p_h \\ &\qquad (i \neq j;\ h \neq i, j), \end{aligned}$$

$$(2.10) \qquad \begin{aligned} U(ij, hk) &= \tfrac{1}{2}p_i + \tfrac{1}{2}p_j + \tfrac{1}{2}(p_i + p_j) + \sum_{l \neq i, j, h, k} (\tfrac{1}{2}p_l + \tfrac{1}{2}p_l) = 1 - p_h - p_k \\ &\qquad (i \neq j;\ h \neq k;\ h, k \neq i, j). \end{aligned}$$

All the possible cases have thus essentially been worked out. For instance, $U(ij, jj)$ and $U(ij, jh)$ can immediately be written down in view of (2.6) and (2.8), respectively.

—To be continued—