61. Probability-theoretic Investigations on Inheritance. X_3. Non-Paternity Concerning Mother-Child-Child Combinations.

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5. Illustrative examples, recessive genes being existent.

General discussions developed in the preceding sections have exclusively concerned genotypes. If, however, phenotypes containing recessive genes are taken as basic unit, the circumstances will become somewhat different. As frequently mentioned, if recessive genes are existent, genotype of an individual cannot necessarily be determined from its phenotype in a unique manner. Corresponding to the fact that a directly observable character is a phenotype, probabilities a posteriori of possible types of father against every given mother-child combination are also to be determined based upon phenotypes, what will be illustrated by various human blood types.

We first consider ABO blood type. Now, given a mother-child combination $(O;O)$, and types except O alone may be possible as type of father, of which the probabilities a priori for O, A, B have to be taken as

$$\bar{O}=r^2, \qquad \bar{A}=p(p+2r), \qquad \bar{B}=q(q+2r),$$

respectively. On the other hand, the mating $O\times O$, $A\times O$, $B\times O$, orders being taken into account, produce a child O with probabilities

$$1, \qquad \frac{r}{p+2r}, \qquad \frac{r}{q+2r},$$

respectively. Hence, due to Bayes' theorem, probabilities a posteriori of types of father being O, A, B are respectively given by

$$Z(O,O;O)=1\cdot\bar{O}\Big/\Big(1\cdot\bar{O}+\frac{r}{p+2r}\cdot\bar{A}+\frac{r}{q+2r}\cdot\bar{B}\Big)=r,$$

$$Z(A,O;O)=\frac{r}{p+2r}\cdot\bar{A}\Big/\Big(1\cdot\bar{O}+\frac{r}{p+2r}\cdot\bar{A}+\frac{r}{q+2r}\cdot\bar{B}\Big)=p,$$

$$Z(B,O;O)=\frac{r}{q+2r}\cdot\bar{B}\Big/\Big(1\cdot\bar{O}+\frac{r}{p+2r}\cdot\bar{A}+\frac{r}{q+2r}\cdot\bar{B}\Big)=q.$$

In similar ways, the remaining probabilities a posteriori will be determined. The results, together with those similarly obtained on

Q and $Q_{q\pm}$ blood types, are tabulated as follows. It is to be noticed that the results on $Qq_{\pm}$ blood type reduce essentially to those on Q blood type by putting $(v_1, v_2)=(v, 0)$ or $(v_1, v_2)=(0, v)$ and unifying q_- and q_+ into q.

Mother	Father / Child	O	A	B	AB
O	O	r	p	q	0
	A	0	$p+r$	0	q
	B	0	0	$q+r$	p
A	O	r	p	q	0
	A	$\frac{r^2(p+r)}{p^2+3pr+r^2}$	$\frac{p(p+r)(p+3r)}{p^2+3pr+r^2}$	$\frac{qr(p+r)}{p^2+3pr+r^2}$	$\frac{pq(p+2r)}{p^2+3pr+r^2}$
	B	0	0	$q+r$	p
	AB	0	0	$q+r$	p
B	O	r	p	q	0
	A	0	$p+r$	0	q
	B	$\frac{r^2(q+r)}{q^2+3qr+r^2}$	$\frac{pr(q+r)}{q^2+3qr+r^2}$	$\frac{q(q+r)(q+3r)}{q^2+3qr+r^2}$	$\frac{pq(q+2r)}{q^2+3qr+r^2}$
	AB	0	$p+r$	0	q
AB	A	$\frac{r^2}{p+r}$	$\frac{p(p+2r)}{p+r}$	$\frac{qr}{p+r}$	$\frac{pq}{p+r}$
	B	$\frac{r^2}{q+r}$	$\frac{pr}{q+r}$	$\frac{q(q+2r)}{q+r}$	$\frac{pq}{q+r}$
	AB	0	$\frac{p(p+r)}{p+q}$	$\frac{q(q+r)}{p+q}$	$\frac{2pq}{p+q}$

Mother	Father / Child	Q	q
Q	Q	$\frac{u(1+2v)}{1+uv}$	$\frac{v^2}{1+uv}$
	q	u	v
q	Q	1	0
	q	u	v

Mother	Father / Child	Q	q_-	q_+
Q	Q	$\frac{u(1+2v)}{1+uv}$	$\frac{v_1(v+v_2)}{1+uv}$	$\frac{v_2^2}{1+uv}$
	q_-	u	$\frac{v^2+v_1v_2}{v+v_2}$	$\frac{v_2^2}{v+v_2}$
	q_+	u	v_1	v_2
q_-	Q	1	0	0
	q_-	u	$\frac{vv_1(v+2v_2)}{v^2+v_1v_2}$	$\frac{vv_2^2}{v^2+v_1v_2}$
	q_+	u	v_1	v_2
q_+	Q	1	0	0
	q_-	u	v	0
	q_+	u	v_1	v_2

Making use of the thus determined probabilities a posteriori, we can deduce the results in these concrete cases, constructing the following tables; probabilities corresponding to (4.2) as well as to (4.21) are listed.

Mother	1st child \ 2nd child	O	A	B	AB	Part. prob. w. r. t. mother and 1st child
O	O	0	$pr^3(q+r)$	$qr^3(p+r)$	—	$r^3(pr+qr+2pq)$
	A	pqr^3	0	$pqr^2(p+r)$	—	$pqr^2(p+2r)$
	B	pqr^3	$pqr^2(q+r)$	0	—	$pqr^2(q+2r)$
A	O	0	0	$\frac{pqr^3(p+r)}{p+2r}$	$\frac{pqr^2(p+r)^2}{p+2r}$	$pqr^2(p+r)$
	A	p^2qr^2	0	$\frac{pqr(p+r)(p^2+3pr+r^2)}{p+2r}$	$\frac{pq(p+r)^2(p^2+3pr+r^2)}{p+2r}$	$\left\{\begin{array}{l}pq(p+r)(p^2+3pr+r^2)\\ \quad +p^2qr^2\end{array}\right.$
	B	$\frac{p^2qr^3}{p+2r}$	0	0	0	$\frac{p^2qr^3}{p+2r}$
	AB	$\frac{p^2qr^2(p+r)}{p+2r}$	0	0	0	$\frac{p^2qr^2(p+r)}{p+2r}$
B	O	0	$\frac{pqr^3(q+r)}{q+2r}$	0	$\frac{pqr^2(q+r)^2}{q+2r}$	$pqr^2(q+r)$
	A	$\frac{pq^2r^3}{q+2r}$	0	0	0	$\frac{pq^2r^3}{q+2r}$
	B	pq^2r^2	$\frac{pqr(q+r)(q^2+3qr+r^2)}{q+2r}$	0	$\frac{pq(q+r)^2(q^2+3qr+r^2)}{q+2r}$	$\left\{\begin{array}{l}pq(q+r)(q^2+3qr+r^2)\\ \quad +pq^2r^2\end{array}\right.$
	AB	$\frac{pq^2r^2(q+r)}{q+2r}$	0	0	0	$\frac{pq^2r^2(q+r)}{q+2r}$
AB	A	—	0	0	$\frac{1}{2}pqr^2(p+q)$	$\frac{1}{2}pqr^2(p+q)$
	B	—	0	0	$\frac{1}{2}pqr^2(p+q)$	$\frac{1}{2}pqr^2(p+q)$
	AB	—	0	0	0	0

$$L_{ABO}=p(1-p)^4+q(1-q)^4+pqr^2(3-r)$$

Mother	1st child \ 2nd child	Q	q	Part. prob. w. r. t. mother and 1st child
Q	Q	0	0	0
	q	0	0	0
q	Q	0	0	0
	q	uv^4	0	uv^4
				$L_Q=uv^4$

Mother	1st child \ 2nd child	Q	q_-	q_+	Part. prob. w. r. t. mother and 1st child
Q	Q	0	0	0	0
	q_-	0	0	0	0
	q_+	0	0	0	0
q_-	Q	0	0	0	0
	q_-	$uvv_1(v^2+v_1v_2)$	0	0	$uvv_1(v^2+v_1v_2)$
	q_+	$uvv_1v_2^2$	0	0	$uvv_1v_2^2$
q_+	Q	0	0	0	0
	q_-	$uvv_1v_2^2$	0	0	$uvv_1v_2^2$
	q_+	uvv_2^3	$v_1v_2^4$	0	$(uv+v_1v_2)v_2^3$
					$L_{Qq\pm}=uv^4+v_1v_2^4$

It will be noticed, as really seen from the tables, that, corresponding to (4.33), every total probability coincides with the one for one-child case also when recessive genes are existent; that is,

$$L_{ABO}=P_{ABO}, \qquad L_{Qq\pm}=P_{Qq\pm}, \qquad L_Q=P_Q.$$

6. Maximizing distributions for J_0.

The distribution maximizing the probability J_0 obtained in § 2 will be determined by a similar procedure as in the preceding chapters.

The probability in case of MN blood type, given in (2.17), i.e.,

$$(6.1) \qquad J_{0MN}=s^2t^2(2-3st)$$

is regarded as a function of the product st alone, ranging over $0\leqq st\leqq 1/4$. The derivative

$$(6.2) \qquad dJ_{0MN}/d(st)=st(4-9st)$$

remains positive for $0<st<1/4$, the maximizing distribution is attained if and only if $st=1/4$ and hence

$$(6.3) \qquad s=t=1/2, \qquad \overline{M}=\overline{N}=1/4, \qquad \overline{MN}=1/2;$$

the maximum being

$$(6.4)\qquad (J_{0MN})^{\max}=5/64=0.0781.$$

In case of general result (2.16), the stationary value

$$(6.5)\qquad (J_0)^{\text{stat}}=\left(1-\frac{1}{m}\right)\left(1-\frac{3}{m}-\frac{1}{m^2}+\frac{15}{m^3}-\frac{31}{2m^4}\right),$$

attained by symmetric distribution

$$(6.6)\qquad p_i=1/m \qquad (i=1, \dots, m),$$

will probably be the actual maximum.

In case of ABO blood type, the probability given in (2.18), i.e.,

$$(6.7)\qquad J_{0ABO}=p^2(1-p)^4+q^2(1-q)^4+\tfrac{1}{2}pqr^2(1+7r^2),$$

is regarded as a function of p and q ($r\equiv1-p-q$). The system of equations $\partial J_{0ABO}/\partial p=\partial J_{0ABO}/\partial q=0$ yields the maximizing distribution

$$(6.8)\qquad \begin{aligned} &p=q=0.2481, \quad r=0.5038;\\ &\bar{O}=0.2538, \quad \bar{A}=\bar{B}=0.3116, \quad \overline{AB}=0.1230; \end{aligned}$$

the common value of p and q is determined as a root quartic equation

$$(6.9)\qquad 174x^4-300x^3+196x^2-57x+6=0.$$

The maximum of (6.7) corresponding to (6.8) is equal to

$$(6.10)\qquad (J_{0ABO})^{\max}=0.0610.$$

In case of Q blood type, we get similarly for (2.19), i.e.,

$$(6.11)\qquad J_{0Q}=u^2v^4;$$

the maximizing distribution and the maximum:

$$(6.12)\qquad u=1/3, \quad v=2/3; \quad \bar{Q}=5/9, \quad \bar{q}=4/9;$$

$$(6.13)\qquad (J_{0Q})^{\max}=16/729=0.0219.$$

In case of $Qq_\pm$ blood type, the probability given in (2.20), i.e.,

$$(6.14)\qquad J_{0Qq\pm}=u^2v^4+(2u+v_1)v_1v_2^4$$

can be regarded as a function of v and v_2 ($u=1-v$, $v_1=v-v_2$). The system of equations determining maximizing distribution becomes

$$(6.15)\qquad \begin{aligned} &0=\partial J_{0Qq\pm}/\partial v=2(1-v)(v^3(2-3v)+v_2^4),\\ &0=\partial J_{0Qq\pm}/\partial v_2=2v_2^3(2v(2-v)-v_2(5-3v_2)); \end{aligned}$$

the roots satisfying $0<v_2<v<1$ being desired. The system (6.15) is written in an equivalent form

$$(6.16)\qquad v_2=\frac{v(100-98v-6v^2+69v^3)}{5(25-24v+12v^2)}, \quad 2v(2-v)=v_2(5-3v_2).$$

By eliminating v_2, an equation of degree seven with respect to v will be obtained. But, the detailed discussion will be omitted. We notice here, remembering (6.12), only an estimation stating that

$$(6.17)\quad \begin{aligned}(J_{0Qq\pm})^{\max} &\geqq (J_{0Q})^{\max} + \operatorname*{Max}_{0\leqq v_2\leqq 2/3}\left(2\cdot\frac{1}{3}+\frac{2}{3}-v_2\right)\left(\frac{2}{3}-v_2\right)v_2^4 \\ &= \frac{16}{729}+\left(\frac{4}{3}-\frac{15-\sqrt{33}}{18}\right)\left(\frac{2}{3}-\frac{15-\sqrt{33}}{18}\right)\left(\frac{15-\sqrt{33}}{18}\right)^4 \\ &= 0.0315.\end{aligned}$$