58. *On the Induced Characters of a Group.*

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This short note is a preliminary report for the theory of induced characters of a group. The detailed proofs will be given elsewhere. The present study is closely related to the papers Brauer [1] and [3].

1. Let $\mathfrak{G}$ be a group of finite order $g=q^a g'$ where q is a prime number and $(g', q)=1$ and let $\mathfrak{Q}$ be a fixed q-Sylow-subgroup of $\mathfrak{G}$. Let $C_1, C_2, \dots, C_n$ be the classes of conjugate elements in $\mathfrak{G}$. Further let $C_1, C_2, \dots, C_h$ be the classes of conjugate elements which contain the elements in $\mathfrak{Q}$. We denote by $Q_1=1, Q_2, \dots, Q_h$ $(Q_i \in \mathfrak{Q})$ a complete system of representatives for the classes C_i $(i=1, 2, \dots, h)$. Let $g_i=g/n_i$ be the number of elements in C_i, so that n_i is the order of the normalizer $\mathfrak{N}(Q_i)$ of Q_i in $\mathfrak{G}$. We set $n_i=q_i n_i'$ where $(n_i', q)=1$. q_i is called the q-part of n_i. Let $\varsigma_1, \varsigma_2, \dots, \varsigma_n$ and $\vartheta_1, \vartheta_2, \dots, \vartheta_m$ be distinct irreducible characters of $\mathfrak{G}$ and $\mathfrak{Q}$. In what follows we shall always take ς_1 and ϑ_1 to be the characters of the 1-representations of $\mathfrak{G}$ and $\mathfrak{Q}$. If ϑ_ν^* is the character of $\mathfrak{G}$ induced from ϑ_ν, then we have the following Frobenius formulas

$$(1) \qquad \begin{cases} \varsigma_\mu(Q) = \sum_\nu r_{\mu\nu}\vartheta_\nu(Q) & \text{(for } Q \text{ in } \mathfrak{Q}) \\ \vartheta_\nu^*(G) = \sum_\mu r_{\mu\nu}\varsigma_\mu(G) & \text{(for } G \text{ in } \mathfrak{G}), \end{cases}$$

where

$$(2) \qquad r_{11}=1, \quad r_{1\nu}=0 \quad (\nu\neq 1).$$

As is well known, the rank of $M=(r_{\mu\nu})$ is h. We can prove, by the similar way as in Brauer [3][1], the following

Lemma 1. *$M=(r_{\mu\nu})$ contains a minor of degree h which is not divisible by q.*

We set

$$R_1 = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1h} \\ r_{21} & r_{22} & \dots & r_{2h} \\ \dots & \dots & \dots & \dots \\ r_{h1} & r_{h2} & \dots & r_{hh} \end{pmatrix}.$$

Then we may assume without restriction that

1) We can somewhat simplify Brauer's original proof.

$$(3) \qquad |R_1| \not\equiv 0 \pmod{q}.$$

We set

$$(4) \qquad M=\begin{pmatrix} R_1 & R_3 \\ R_2 & R_4 \end{pmatrix}=(R,\ *), \quad R=\begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

Since the rank of M is h, we have

$$(5) \qquad \begin{pmatrix} R_3 \\ R_4 \end{pmatrix}=RB,$$

where

$$(6) \qquad B=\begin{pmatrix} b_{h+1,1} & b_{h+2,1} & \cdots & b_{m,1} \\ b_{h+1,2} & b_{h+2,2} & \cdots & b_{m,2} \\ \cdots & \cdots & \cdots & \cdots \\ b_{h+1,h} & b_{h+2,h} & \cdots & b_{m,h} \end{pmatrix}.$$

By (3), we see that the coefficients $b_{h+\kappa,\lambda}$ are the rational numbers whose denominators are prime to q. Further from (2)

$$(7) \qquad b_{h+\kappa,1}=0 \qquad (\kappa=1, 2, \dots, m-h).$$

Since $M=R(I, B)$, we obtain

$$(8) \qquad (\varsigma_\mu(Q_i))=R(I,\ B)(\vartheta_\nu(Q_i))=R(\tilde{\vartheta}_\lambda(Q_i)),$$

where

$$(9) \qquad \tilde{\vartheta}_\lambda=\vartheta_\lambda+\sum_{\kappa=1}^{m-h} b_{h+\kappa,\lambda}\vartheta_{h+\kappa} \qquad (\lambda=1, 2, \dots, h).$$

In particular, (7) yields

$$(10) \qquad \tilde{\vartheta}_1=\vartheta_1.$$

From (8) we have

$$(11) \qquad \varsigma_\mu(Q)=\sum_{\lambda=1}^{h} r_{\mu\lambda}\tilde{\vartheta}_\lambda(Q) \qquad (\text{for } Q \text{ in } \mathfrak{Q}).$$

Lemma 2. *If Q and Q' in $\mathfrak{Q}$ are conjugate in $\mathfrak{G}$, then $\tilde{\vartheta}_\lambda(Q)=\tilde{\vartheta}_\lambda(Q')$.*

We obtain the following important

Theorem 1. *If $\tilde{\Theta}=(\tilde{\vartheta}_\lambda(Q_i))$, then*

$$|\tilde{\Theta}|^2=q_1q_2\dots q_h/v,$$

where v is a rational integer which is prime to q.

We set

$$(12) \qquad Z=(\varsigma_\mu(Q_i)), \quad \Theta^*=(\vartheta_\lambda{}^*(Q_i)),$$

$\mu=1, 2, \dots, n : \lambda,\ i=1, 2, \dots, h$. Then (1) and (8) yield

$$(13) \qquad \Theta^*=R'Z=R'R\tilde{\Theta}.$$

If we set $W=R'R=(w_{\kappa\lambda})$, then $w_{\kappa\lambda}=\sum r_{\mu\kappa}r_{\mu\lambda}=w_{\lambda\kappa}$ and

$$\vartheta_\kappa^*(Q)=\sum_{\lambda=1}^{h} w_{\kappa\lambda}\tilde{\vartheta}_\lambda(Q). \tag{14}$$

Now we obtain the following theorems.

Theorem 2. $\sum_\lambda \vartheta_\lambda^*(Q_i)\,\tilde{\vartheta}_\lambda(Q_j^{-1})=n_i\delta_{ij}$, *where n_i is the order of the normalizer $\mathfrak{N}(Q_i)$ of Q_i in $\mathfrak{G}$.*

Theorem 3. $\sum_i g_i\tilde{\vartheta}_\kappa^*(Q_i)\tilde{\vartheta}_\lambda(Q_i^{-1})=g\delta_{\kappa\lambda}$.

If we take $\lambda=1$, then from (10) we have

$$\sum_{Q^*}\vartheta_\kappa^*(Q^*)=\begin{cases} g & \text{for } \kappa=1 \\ 0 & \text{for } \kappa\neq 1,\end{cases} \tag{15}$$

where Q^* ranges over all elements of $\mathfrak{G}$ whose orders are powers of q.

Since $|Z'Z|=|\tilde{\Theta}'W\tilde{\Theta}|=\pm n_1n_2\dots n_h$, we have by Theorem 1

$$|W|\not\equiv 0 \pmod{q}. \tag{16}$$

We can distribute the irreducible characters ς_μ of $\mathfrak{G}$ into blocks with respect to $\mathfrak{Q}$[2]. This will be reserved for a subsequent paper.

2. Let $A_0=1, A_1, A_2, \dots, A_k$ be a maximal system of elements of $\mathfrak{G}$ such that A_i, A_j are not conjugate for $i\neq j$ and the order of each A_i is prime to q. Let $\mathfrak{N}_i$ be the normalizer of A_i in $\mathfrak{G}$ and let $\mathfrak{Q}_i$ be a q-Sylow-subgroup of $\mathfrak{N}_i$. A full system $\sum$ of elements of $\mathfrak{G}$ representing the different classes of conjugate elements can be obtained in the following manner: Let $Q_{i,1}, Q_{i,2}, \dots, Q_{i,h(i)}$ $(Q_{i,j}\in\mathfrak{Q}_i)$ represent the different classes of conjugate elements in $\mathfrak{N}_i$, in which the orders of the elements are powers of q. Then $\sum$ consists of the elements $A_iQ_{i,1}, A_iQ_{i,2}, \dots, A_iQ_{i,h(i)}$ for $i=0,1,2,\dots,k$. Let us denote by $n_{i,j}$ the order of the normalizer $\mathfrak{N}(A_iQ_{i,j})$ of $A_iQ_{i,j}$ in $\mathfrak{G}$. Then the order of the normalizer $\mathfrak{N}(Q_{i,j})$ in $\mathfrak{N}_i$ is equal to $n_{i,j}$.

We denote by $\vartheta_{i,\nu}(\nu=1,2,\dots,m(i))$ the irreducible characters of $\mathfrak{N}_i$. Then we obtain by the similar way as in Brauer [2][3]

$$\varsigma_\mu(A_iQ_{i,j})=\sum_{\lambda=1}^{h(i)} r^i_{\mu\lambda}\,\tilde{\vartheta}_{i,\lambda}(Q_{i,j}), \tag{17}$$

where $r^i_{\mu\lambda}$ are algebraic integers and $\tilde{\vartheta}_{i,\lambda}$ have the same significance for $\mathfrak{N}_i$ as $\tilde{\vartheta}_\lambda$ for $\mathfrak{G}$. We arrange these numbers $r^i_{\mu\lambda}$ for a fixed i in form of a matrix $R^i=(r^i_{\mu\lambda})$ and set

$$R^*=(R^0, R^1, \dots, R^k), \qquad R^0=R. \tag{18}$$

According to (17) we have a formula $(\varsigma_\mu(A_iQ_{i,j}))=R^*V$. The matrix V

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2) See Osima [5].

3) Cf. Brauer [2] p. 927.

breaks up completely into the matrices $(\tilde{\vartheta}_{i,\lambda}(Q_{i,j}))$ $(i=0, 1, 2, \dots, k)$. Since $(\varsigma_\mu(A_iQ_{i,j}))$ is non-singular, so is R^*. Let us denote by $q_{i,j}$ the q-part of $n_{i,j}$. Then, by Theorem 1

(19) $$| V |^2 = \prod_{i=0}^{k} (\prod_{j=1}^{h(i)} q_{i,j}/v_i).$$

Here $v_0=v$ and $(v_i, q)=1$. Hence we obtain

(20) $$| R^* | \not\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{q}),$$

where $\mathfrak{q}$ is a prime ideal which divides q.

Let $\vartheta^*_{i,\lambda}$ be the character of $\mathfrak{N}_i$ induced from the irreducible character $\vartheta_{i,\lambda}$ of $\mathfrak{Q}_i$. We denote by $\bar{a}$ the number conjugate complex to a. Then, from

$$\sum_\mu \varsigma_\mu(A_iQ_{i,s}) \overline{\varsigma_\mu(A_jQ_{j,t})} = n_{i,s}\, \delta_{i,j}\, \delta_{s,t},$$

we can derive

(21) $$\sum_\mu \bar{r}^j_{\mu\lambda}\, \varsigma_\mu(A_iQ_{i,s}) = \vartheta^*_{i,\lambda}(Q_{i,s})\, \delta_{ij}.$$

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(22) $$\sum_\mu r^i_{\mu\kappa}\, \bar{r}^j_{\mu\lambda} = w^i_{\kappa\lambda}\, \delta_{ij},$$

where $w^i_{\kappa\lambda}$ have the same significance for $\mathfrak{N}_i$ as $w_{\kappa\lambda}$ for $\mathfrak{G}$.

The group $\mathfrak{H}_i=\{A_i, \mathfrak{Q}_i\}$ generated by A_i and $\mathfrak{Q}_i$ is a direct product: $\mathfrak{H}_i=\{A_i\}\times\mathfrak{Q}_i$. An irreducible character $\psi^{(i)}_\rho$ of $\mathfrak{H}_i$ is the product of an irreducible character $\chi_{i,\alpha}$ of $\{A_i\}$ and an irreducible character $\vartheta_{i,\nu}$ of $\mathfrak{Q}_i$:

(23) $$\psi^{(i)}_\rho(A_iQ_{i,j}) = \chi_{i,\alpha}(A_i)\vartheta_{i,\nu}(Q_{i,j}).$$

We denote by $(\chi_{i\alpha}\vartheta_{i,\nu})^*$ the character of $\mathfrak{G}$ induced from the character $\chi_{i,\alpha}\vartheta_{i,\nu}$. Let

(24) $$\varsigma_\mu(A_iQ_{i,j}) = \sum_\nu \sum_\alpha r^i_{\alpha\mu\nu}\, \chi_{i,\alpha}(A_i)\, \vartheta_{i,\nu}(Q_{i,j}).$$

Then

(25) $$(\chi_{i,\alpha}\vartheta_{i,\nu})^* = \sum_\mu r^i_{\alpha\mu\nu}\, \varsigma_\mu.$$

We have from (17) and (24)

(26) $$r^i_{\mu\lambda} = \sum_\alpha r^i_{\alpha\mu\lambda}\, \chi_{i,\alpha}(A_i) \qquad (\lambda=1, 2, \dots, h(i)).$$

From (20) and (26) we can prove directly Theorem 1 in Brauer [3].

3. Above arguments are also applicable to the theory of modular characters of $\mathfrak{G}$ for a prime $p\neq q$. Let $\varphi_1, \varphi_2, \dots, \varphi_l$ be distinct absolutely irreducible modular characters of $\mathfrak{G}$ and let $\eta_1, \eta_2, \dots, \eta_l$ be the characters of indecomposable constituents of the regular representation of $\mathfrak{G}$ (mod p). Let $C_1, C_2, \dots, C_l$ be the classes of conjugate elements in $\mathfrak{G}$, in which the orders of the elements are prime to p. We denote by $H_1, H_2, \dots, H_l$ a complete system of representatives for the classes C_i $(i=1, 2, \ldots, l)$. We may assume that $H_i=Q_i$ $(i=1, 2, \ldots, h)$. We have [4]

(27)
$$\begin{cases} \varphi_\kappa(Q_i) = \sum_{\nu=1}^{m} s_{\kappa\nu}\vartheta_\nu(Q_i) \\ \vartheta_\nu^*(H_j) = \sum_{\kappa=1}^{l} s_{\kappa\nu}\eta_\kappa(H_j) . \end{cases}$$

Using (27) we obtain from Theorem 2

(28)
$$\sum_{\kappa=1}^{l}\eta_\kappa(H_j)\,(\sum_{\lambda=1}^{h} s_{\kappa\lambda}\,\tilde{\vartheta}_\lambda(Q_i^{-1}))=\begin{cases} n_i & (\text{for } H_j=Q_i) \\ 0 & (\text{for } H_j\neq Q_i). \end{cases}$$

On the other hand, we have

(29)
$$\sum_{\kappa=1}^{l}\eta_\kappa(H_j)\varphi_\kappa(Q_i^{-1})=\begin{cases} n_i & (\text{for } H_j=Q_i) \\ 0 & (\text{for } H_j\neq Q_i). \end{cases}$$

Since $\eta_1, \eta_2, \ldots, \eta_l$ are linearly independent, we have from (28) and (29)

(30)
$$\varphi_\kappa(Q_i)=\sum_{\lambda=1}^{h} s_{\kappa\lambda}\,\tilde{\vartheta}_\lambda(Q_i).$$

We denote by $d_{\mu\kappa}$ the decomposition numbers of $\mathfrak{G}$ for p:

(31)
$$\varsigma_\mu(H_j) = \sum_\kappa d_{\mu\kappa}\varphi_\kappa(H_j).$$

We have from (11), (30), and (31)

(32)
$$r_{\mu\lambda} = \sum_\kappa d_{\mu\kappa}s_{\kappa\lambda} ,$$

or in matrix form

(33)
$$R = DS,$$

where $D=(d_{\mu\kappa})$ and $S=(s_{\kappa\lambda})$. Let C be the matrix of Cartan invariants of $\mathfrak{G}$. Since $C=D'D$, we obtain

(34)
$$W=R'R=S'D'DS=S'CS.$$

Let $A_0, A_1, \ldots, A_k$ have the same significance as in §2. Then we may assume that $A_0, A_1, \ldots, A_t$ are a maximal system of elements of $\mathfrak{G}$ such that A_i, A_j are not conjugate for $i\neq j$ and the order of each A_i is prime to p and q. We obtain by the similar way as in §2

(35)
$$\varphi_\kappa(A_iQ_{i,j}) = \sum_{\lambda=1}^{h(i)} s_{\kappa\lambda}^i\,\tilde{\vartheta}_{i,\lambda}\,(Q_{i,j}),$$

where $s_{\kappa\lambda}^i$ are algebraic integers. We set $S^i=(s_{\kappa\lambda}^i)$ and

(36)
$$S^*=(S^0, S^1, \ldots, S^t), \qquad S^0=S.$$

Then we have

(37)
$$\Phi=(\varphi_\kappa(A_iQ_{ij}))=S^*U.$$

4) See Brauer and Nesbitt [4] § 26.

The matrix U breaks up completely into the matrices $(\tilde{\vartheta}_{i,\lambda}(Q_{i,j}))$ $(i=0, 1, 2, \dots, t)$.
By Theorem 1

$$|U|^2=\prod_{i=0}^{t}\Big(\prod_{j=1}^{h(i)} q_{i,j}/v_i\Big), \qquad (v_i, q)=1. \tag{38}$$

Since $|\Phi|^2|C|=\prod_i (\prod_j n_{i,j})$[5], we have $|S^*|\not\equiv 0 \pmod q$. Further from (38) we see that $|C|\not\equiv 0 \pmod q$. This, combined with $(|\Phi|, p)=1$, yields

Theorem 4. *The determinant $|c_{\kappa\lambda}(p)|$ of the matrix of Cartan invariants of $\mathfrak{G}$ is a power of p*[6].

References.

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[5] M. Osima: On the representations of groups of finite order, Math. J. Okayama Univ., 1 (1952).

5) See Brauer and Nesbitt [4].

6) Brauer [1] Theorem 1.