78. On Functions Harmonic in a Circle, with Special Reference to Poisson Representation.

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1. We consider a family of functions harmonic in the unit circle of the $z=re^{i\theta}$-plane. It is well known that the Dirichlet problem, i. e. the first boundary value problem on harmonic functions, for the unit circle is solved by the *Poisson integral formula*

$$u_0(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}f(\varphi)d\varphi,$$

in the sense that, $f(\varphi)$ being prescribed as any boundary value function integrable for $0\leqq\varphi<2\pi$, the function $u_0(z)$ defined by the formula is harmonic in $|z|<1$ and tends to $f(\varphi)$ almost everywhere in $0\leqq\varphi<2\pi$ as z tends to $e^{i\varphi}$ along a Stolz path.

The Poisson formula, especially in case of bounded boundary values, is characterized by its special behavior that, if $f(\varphi)$ is restricted by $f_s\leqq f(\varphi)\leqq f_a$ for $0\leqq\varphi<2\pi$, then the function $u_0(z)$ associated to $f(\varphi)$ by the formula submits to the same restriction $f_s\leqq u_0(z)\leqq f_a$ in $|z|<1$.

However, in case of unbounded boundary values, the circumstance becomes somewhat complicated. Although, for instance, a function

$$\frac{1-|z|^2}{|e^{i\varphi}-z|^2}\equiv\Re\frac{e^{i\varphi}+z}{e^{i\varphi}-z}$$

is harmonic in $|z|<1$ for any fixed φ and has boundary values vanishing everywhere except at $e^{i\varphi}$ alone, it must once be excluded to add a linear combination of such functions with various φ's or of unbounded functions of analogous character. The uniqueness of the solution of Dirichlet problem in the proper sense can thus be verified. The Poisson formula displays its effect concerning Dirichlet problem essentially in a range of the bounded harmonic functions, while it may be suitably generalized to a certain extent.

In the present Note, we shall discuss the problems on Poisson integral from the latter version, especially those characterizing the family of functions representable by Poisson integral.

2. We begin with an extremal property of Poisson integral for functions bounded in one side.

Theorem 1. *Let $f(\varphi)$ be a function integrable for $0\leqq\varphi<2\pi$ and* *let* $\mathfrak{U}$ *denote a family of all functions which are harmonic and bounded below in* $|z|<1$ *and tend to* $f(\varphi)$ *almost everywhere as* z *tends to* $e^{i\varphi}$ *along a Stolz path. Then, the function* $u_0(z)$ *associated to* $f(\varphi)$ *by Poisson integral is the smallest one within the family* $\mathfrak{U}$. *More precisely stated, any admissible function* $u(z)\in\mathfrak{U}$ *satisfies an inequality*

$$u_0(z)\leqq u(z)$$

throughout $|z|<1$. *Further, if the equality sign appears at a point in* $|z|<1$, *then* $u(z)$ *must coincide with* $u_0(z)$ *identically.*

Proof. It is enough to prove the inequality only at the origin, since we have only to combine a linear transformation mapping the unit circle onto itself. By assumption, $u(re^{i\varphi})$ is bounded below and tends to $f(\varphi)$ almost everywhere as $r\to 1-0$. Hence, in view of a theorem due to Fatou, we get

$$\int_0^{2\pi} f(\varphi)d\varphi \leqq \varliminf_{r\to 1-0}\int_0^{2\pi} u(re^{i\varphi})d\varphi.$$

But, again by assumption and in view of the mean value theorem, we have

$$u_0(0)=\frac{1}{2\pi}\int_0^{2\pi} f(\varphi)d\varphi, \qquad u(0)=\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\varphi})d\varphi$$

for any r with $0\leqq r<1$, whence we conclude the desired inequality $u_0(0)\leqq u(0)$. The last part of the statement is an immediate consequence of the maximum principle.

If $u(z)$ is assumed to be *bounded above* instead of below, then we can conclude that an opposite inequality

$$u_0(z)\geqq u(z)$$

holds good. This fact, combined with theorem 1, implies incidentally the uniqueness of the solution of Dirichlet problem for a circle in case of bounded boundary values.

3. Now, under the assumptions of theorem 1, the difference $u(z)-u_0(z)$ remains non-negative in $|z|<1$. We can, moreover, express this difference in a quantitative manner, sharpening theorem 1 as follows.

Theorem 2. *Under the same assumptions as in theorem* 1, *we may write*

$$u(z)=u_0(z)+\int_{e_0}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}d\mu(\varphi),$$

where $\mu(e)$ *denotes a non-negative additive set function defined for every Borel set* e *contained in* $C: 0\leqq\varphi<2\pi$ *and* e_0 *is a Borel null set, both depending on* $u(z)$.

Proof. Applying to the difference $u(z)-u_0(z)$ which is harmonic and remains, in view of theorem 1, non-negative in $|z|<1$, a representation by Radon-Stieltjes integral, known as *Herglotz represen-*

tation, we get

$$u(z)-u_0(z)=\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}d\mu(\varphi),$$

$\mu(e)$ being a non-negative additive set function with $\mu(C)=u(0)-u_0(0)$. Now, according to the decomposition theorem of Lebesgue, the set function $\mu(e)$ can be decomposed in the form

$$\mu(e)=\Phi(e)+\theta_{e_0}(e),$$

where $\Phi(e)$ is a non-negative absolutely continuous additive set function and $\theta_{e_0}(e)\equiv\mu(e_0\frown e)$ is a singular one, e_0 being a Borel null set. Hence, we obtain

$$u(z)-u_0(z)=\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}d\Phi(\varphi)+\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}d\theta_{e_0}(\varphi)$$
$$=\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}\Phi'(\varphi)d\varphi+\int_{e_0}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}d\mu(\varphi),$$

where $\Phi'(\varphi)$ denotes the derivative, being existent almost everywhere, of $\Phi(e)$. Now, the second term in the right-hand side of the last expression originating in a singular part remains non-negative throughout $|z|<1$, while the first term, being a Poisson integral, tends to a non-negative boundary function $2\pi\Phi'(\varphi)$ almost everywhere. But, by assumption, the left-hand side has the boundary values vanishing almost everywhere. Thus, we conclude that $\Phi'(\varphi)$ vanishes almost everywhere and further, in view of the absolute continuity, $\Phi(e)$ must vanish out. Our assertion has thus been proved.

Since $\mu(e)$ is a non-negative set function, theorem 1 is an immediate consequence of theorem 2 just proved. By the way, theorem 2 implies a condition of representability of $u(z)$ by Poisson integral; namely, the condition requiring that the set e_0 be empty, i. e. $\mu(e)$ be absolutely continuous. This fact will be stated below in theorem 4 with a slight generalization. On the other hand, the above argument shows incidentally that *the integral*

$$\int_{e_0}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}d\mu(\varphi)$$

with $\mu(e)$ *associated to* $u(z)-u_0(z)$ *and hence also to* $u(z)$ *represents a function harmonic in* $|z|<1$ *and with boundary values vanishing almost everywhere.*

If $u(z)$ is assumed to be *bounded above* instead of below, then we can derive a representation

$$u(z)=u_0(z)-\int_{e_0}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}d\mu(\varphi)$$

with $\mu(e)$ and e_0 of analogous character as before.

4. Theorem 2 can be further generalized to case where $f(\varphi)$ is bounded neither below nor above, in the form stated as follows.

Theorem 3. *Let $f(\varphi)$ be a function integrable for $0 \leqq \varphi < 2\pi$ and $u_0(z)$ be the function associated to $f(\varphi)$ by Poisson integral. Let $u(z)$ be any function which is harmonic in $|z|<1$ and tends to $f(\varphi)$ almost everywhere as z tends to $e^{i\varphi}$ along a Stolz path and for which the integral*

$$\int_0^{2\pi} |u(re^{i\varphi})| d\varphi$$

is bounded in r with $0 \leqq r < 1$. Then, we may write

$$u(z) = u_0(z) + \int_{e_1} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} d\mu_1(\varphi) - \int_{e_2} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} d\mu_2(\varphi),$$

where $\mu_1(e)$ and $\mu_2(e)$ denote non-negative additive set functions defined for every Borel set e belonging to $C: 0 \leqq \varphi < 2\pi$, and e_1 and e_2 are disjoint Borel null sets; these set functions and null sets depending on $u(z)$.

Proof. The assumption that the integral of $|u(re^{i\varphi})|$ over C is bounded, implies again the applicability of Herglotz representation, yielding

$$u(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} d\mu(\varphi),$$

while here an additive set function $\mu(e)$ is of bounded value-sum, that is, its corresponding point function, $\mu(\varphi)$ say, is of bounded variation. Applying further a Jordan decomposition of $\mu(e)$ into a difference of non-negative set functions:

$$\mu(e) = \mu_1(e) - \mu_2(e),$$

we get

$$u(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} d\mu_1(\varphi) - \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} d\mu_2(\varphi).$$

The right-hand side of the last expression is thus a difference of the expressions just of the form considered in theorem 2, and hence a similar argument will also be valid. In fact, decomposing the $\mu_\nu(e)$ ($\nu=1, 2$) according to Lebesgue manner into

$$\mu_\nu(e) = \Phi_\nu(e) + \theta_{e_\nu}(e),$$

the above expression becomes

$$u(z) = \int_0^{2\pi} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} d\Phi(\varphi) + \int_{e_1} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} d\mu_1(\varphi) - \int_{e_2} \frac{1-|z|^2}{|e^{i\varphi}-z|^2} d\mu_2(\varphi)$$

with $\Phi(e) = \Phi_1(e) - \Phi_2(e)$. Since the second as well as the third terms in the right-hand side possess, as seen from an above notice, the boundary values vanishing almost everywhere, the boundary values of the right-hand side coincide with $2\pi\Phi'(\varphi)$ almost everywhere, while those of the left-hand side are, by assumption, equal to $f(\varphi)$ almost everywhere. Hence, we have $f(\varphi)=2\pi\Phi'(\varphi)$ almost everywhere, whence follows the desired representation.

Finally, it would be noticed that theorem 2 infers a condition of representability by Poisson integral, stated as follows.

Theorem 4. *A function $u(z)$ harmonic in $|z|<1$ and tending almost everywhere to an integrable boundary function $f(\varphi)$ can be represented in the form*

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}f(\varphi)d\varphi,$$

if and only if the integral

$$\int_0^{2\pi}|u(re^{i\varphi})|d\varphi$$

is bounded in r with $0\leqq r<1$ and the set function $\mu(e)$ contained in its Herglotz representation is absolutely continuous.

Proof. The sufficiency of the condition is obvious. Its necessity can be assured in the following way. In fact, let $u(z)$ be represented by the Poisson integral associated to $f(\varphi)$. We introduce by

$$u^{\pm}(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\varphi}-z|^2}\frac{1}{2}(f(\varphi)\pm|f(\varphi)|)d\varphi$$

two functions $u^+(z)$ and $u^-(z)$ harmonic in $|z|<1$ and with definite but mutually opposite signs. Then, we get

$$u(z)=u^+(z)+u^-(z)$$

and hence

$$\int_0^{2\pi}|u(re^{i\varphi})|d\varphi\leqq\int_0^{2\pi}(u^+(re^{i\varphi})-u^-(re^{i\varphi}))d\varphi=2\pi(u^+(0)-u^-(0)).$$

On the other hand, the functions $u^+(z)-u(z)$ and $u(z)-u^-(z)$, both being non-negative in $|z|<1$, are representable by means of Poisson integral and hence, in view of theorem 2, the additive set functions contained in their Herglotz representations must have the singular parts in Lebesgue decomposition vanishing identically. Consequently, the set function belonging to

$$u(z)-u^-(z)-(u^+(z)-u(z))\equiv u(z)$$

must also be absolutely continuous.

Addendum. It should be cited that A. J. Lohwater has dealt with a related problem under slightly different assumptions in his recent paper, A uniqueness theorem for a class of harmonic functions, Proc. Amer. Math. Soc. **3** (1952), 278–279, which has been brought to the author's attention while the present Note was in press.