75. On Rings of Operators of Infinite Classes. II.

By Haruo SUNOUCHI.

Mathematical Institute, Tôhoku University, Sendai.

(Comm. by Z. SUETUNA, M.J.A., July 12, 1952.)

In the previous paper [5], we have extended the notion of the $\natural$-operation, introduced by Dixmier [1], to the rings of operators of the infinite classes. But the statements of the last section of [5] are not complete, therefore we will precisely discuss them with some modifications. Especially, we shall clarify the relation between the finiteness and the E-finiteness of a projection. By the way, we obtain a functional characterisation of the abelian rings of operators, which is a generalisation of von Neumann's one in separable cases [3; Theorem 6].

1. Firstly we shall remember some definitions. Let $\boldsymbol{M}$ be a ring of operators in a Hilbert space $\boldsymbol{H}$, and denote the center by $\boldsymbol{M}^{\natural}$. A projection $P \in \boldsymbol{M}$ is called *finite* if, for any projection $Q \in \boldsymbol{M}$, $P \sim Q \leqq P$ implies $Q=P$, and *infinite* if this is not the case. If the unit element $I \in \boldsymbol{M}$ is finite, then we say $\boldsymbol{M}$ is of a *finite class*, and otherwise $\boldsymbol{M}$ is of an *infinite class*. As remarked in [5], any ring of operators $\boldsymbol{M}$ is decomposed into the direct sum of three rings of operators, $\boldsymbol{M}^{f}$, $\boldsymbol{M}^{i}$, and $\boldsymbol{M}^{pi}$, say; $\boldsymbol{M}^{f}$ is of the finite class, $\boldsymbol{M}^{i}$ is the one, in which every central projection is infinite but there exists a finite projection in it, and $\boldsymbol{M}^{pi}$ is in the other case. We say $\boldsymbol{M}^{pi}$ is *of the purely infinite class*. For a while, we shall assume that $\boldsymbol{M}=\boldsymbol{M}^{i}$, because, in $\boldsymbol{M}^{f}$, the Dixmier theory is applicable, and in $\boldsymbol{M}^{pi}$, our arguments are not available.

By a *central envelope* of a finite projection E we mean the central projection Z, which is the least upper bound of $F \in \boldsymbol{M}$ equivalent to E. Then there is a system of finite projections $E_\alpha \in \boldsymbol{M}$, such that each E_α has no comparable part to others and the corresponding central envelopes Z_α span the unit I. Denote $E=\sum \oplus E_\alpha$ for this system.

Lemma 1.1. *Let E_α be the finite projections in $\boldsymbol{M}$, which have no comparable parts to each other, then $E=\sum \oplus E_\alpha$ is also finite.*

Proof. The assumption is equivalent to that the corresponding central envelopes Z_α are mutually orthogonal. Let $Z=\sum \oplus Z_\alpha$, then Z is obviously the central envelope of E. Any projection $F \in \boldsymbol{M}_{(Z)}$[1] is written in the form: $F=\sum \oplus F_\alpha$, where $F_\alpha=FZ_\alpha$. Naturally

1) $\boldsymbol{M}_{(E)}$ denotes the set of all $A_{(E)}=EA=AE$, $A \in \boldsymbol{M}$.

$E_\alpha = EZ_\alpha$. Assume $E \sim F \leqq E$, then $E_\alpha \sim F_\alpha \leqq E_\alpha$; by the fact that each E_α is finite, $F_\alpha = E_\alpha$ or we have $F = E$.

Therefore, if we take the above mentioned $E = \sum \oplus E_\alpha$, then

Corollary 1.1. *In $\boldsymbol{M}$, there exists a finite projection $E \in \boldsymbol{M}$, such that its central envelope is the unit element I.*

We shall call this E the *generalised unit element* of $\boldsymbol{M}$, and in the sequel we discuss $\boldsymbol{M}$ on the basis of this generalised unit E.

Lemma 1.2. *Any projection $P \in \boldsymbol{M}$ can be written in the following form with respect to the generalised unit E:*

$$(1) \qquad P = \sum_{\alpha \in A_1} \oplus E_1^\alpha \oplus F_1 \oplus \sum_{\alpha \in A_2} \oplus E_2^\alpha \oplus F_2 \oplus \cdots \oplus \sum_{\alpha \in A_\mu} \oplus E_\mu^\alpha,$$

where each member is mutually orthogonal and $E_\mu^\alpha \lesssim E$, $E_\mu^\alpha \sim E_\mu^\beta$, $E_\nu^\alpha \prec E_\mu^\alpha$ $(\nu > \mu)$; $F_\mu \prec E_\mu^\alpha$ and F_μ has no comparable parts with the remainders.

Proof. As $I = Z_E$, there exists a projection $F \sim E$ such that $PF \neq 0$. To this F and P, apply the Theorem 6 of [1], then there exist:

two projections $Z_1 \in \boldsymbol{M}^\natural$ and $Z_1' = I \ominus Z_1 \in \boldsymbol{M}^\natural$;

three mutually orthogonal projections F_1, P_1, $\bar{F}$, contained in Z_1;

three mutually orthogonal projections F_2, P_2, $\bar{P}$, contained in Z_1';

such that $F_1 \sim P_1$, $F_2 \sim P_2$, $F = \bar{F} \oplus F_1 \oplus F_2$, and $P = P_1 \oplus P_2 \oplus \bar{P}$. By the construction of this decomposition, it is easily seen that $F_1 \oplus F_2$ is the maximal part equivalent to P, therefore $P \lesssim E$ implies $\bar{F} = 0$. Then we repeat the above decomposition to $P \ominus (P_1 \oplus P_2)$. After some times of repetitions, say A_1, it may come out that $\bar{F} \neq 0$. Then we have $P = \sum_{\alpha \in A_1} \oplus E_1^\alpha \oplus F_1 \oplus P_1$, where $E_1^\alpha \sim E$ and F_1 has no comparable part with P_1, by the construction of the above decomposition. The Z_1' is the central envelope of the $E_2 \sim F_2$, $E_2 < E$, therefore in this part we can repeat the above argument to the P_1, and finally we have $\bar{P} = 0$ by the transfinite induction. This completes the proof.

Thus we know that there is a central decomposition of the unit I corresponding to the expression (1) such that $I = \sum \oplus Z_\mu$, and P is written in the form.

$$(2) \qquad P = \sum \oplus Z_\mu P, \text{ where } Z_\mu P = \sum_{\alpha \in A_\mu} \oplus Z_\mu E_\mu^\alpha \oplus Z_\mu F_\mu.$$

Clearly all $Z_\mu E_\mu^\alpha = E_\mu^\alpha$ are equivalent, and $Z_\mu F_\mu = F_\mu$.

In the previous paper, a projection $P \in \boldsymbol{M}$ is called *E-finite* if μ end with some finite numbers and each A_μ is also finite. If we say in the expression (2), a projection $P \in \boldsymbol{M}$ is *E-finite* if there exist the central projections Z_i such that $I=\sum_{i=1}^{m}\oplus Z_i$ and PZ_i can be written in the form: $PZ_i=\sum_{j=1}^{n(i)}\oplus E_i^j$, $E_i^j\sim E_i^k$, $E_i^j\prec E$.

Moreover we have the following.

Theorem 1. *A projection $P\in \boldsymbol{M}$ is finite if and only if, in the expression (2), μ end with at most countable numbers, $I=\sum_{1\leq\mu<\infty}\oplus Z_\mu$, and PZ_i is E-finite*[2].

Proof. The necessity depends on the following facts: that, in Z_∞, $PZ_\infty=\sum_{i=1}^{\infty}E_\infty^i$, $E_\infty^i\sim E_\infty^1$, and that any projection containing an infinite projection is also infinite. The sufficiency follows by Lemma 1.1 and by the fact that any E-finite projection is finite.

2. Let E be the generalised unit element of $\boldsymbol{M}$. We will now extend the $\natural$-operation to any E-finite projections. For the operator $A\in \boldsymbol{M}_{(E)}$, we have already defined the operation $A^\natural$ in [5; Theorem 1].

Let $Q\in \boldsymbol{M}$ be a central projection, and let $E'=EQ$, then Q is obviously the central envelope of E'. For any operator $A\in \boldsymbol{M}_{(E')}$, we can define an operation $A^{\natural'}$ with respect to E'. But we obtain the following

Lemma 2.1. *For any operator $A\in \boldsymbol{M}_{(E')}$, $A^{\natural'}=A^\natural$.*

Now let a projection $E_1\sim E$, then we can define the operation $\natural_1$ for $\boldsymbol{M}_{(E_1)}$ by the similar way to $\boldsymbol{M}_{(E)}$. But we obtain

Lemma 2.2. *Let the projections $P_1\sim P$, $P_1\leqq E_1$, $P\leqq E$, and let $E_1\sim E$, then $P_1^{\natural_1}=P^\natural$.*

These lemmas follow easily or directly by [5; Theorem 3][3].

By these lemmas, in the expression (1) of an E-finite projection P given in Lemma 1.2, denote the central envelope of each E_i^1 by Q_i, and put

$$(3)\qquad P^\natural=N_1Q_1+F_1^\natural+\dots+N_nQ_n+F_n^\natural,$$

where $N_i=A_i$, then we obtain the operation $\natural$ for any E-finite projection P. Moreover, we remark that

$$(4)\qquad \|P^\natural\|\leqq(\sum_{i=1}^{n}N_i+n)\|P\|,$$

because in $\boldsymbol{M}_{(E_1)}$, $E_1\sim E$, the $\natural$-operation is uniformly continuous (see [5; Theorem 1]).

If an operator $A\in \boldsymbol{M}$ is contained in some E-finite projection P, that is, $AP=PA=A$, then A is called *E-finite*. Now let the pectral decomposition of A be $\int\lambda dP_\lambda$, (as be well-known, it is suf-

2) These circumstances will be also discussed in § 3.

3) It should be noted here that, in the lemma 2 of [5], $E^\natural$ has no inverse $(E^\natural)^{-1}$ $\boldsymbol{M}^\natural{}_{(Z)}$, but almost everywhere in the sense of § 3. But the results of [5] remain true.

ficient to consider only the self-adjoint A), then A is the uniform limit of the form: $A_\varepsilon=\sum_{i=1}^{m}\lambda_i(P_{\lambda_i}-P_{\lambda_{i-1}})$[4]. Clearly every $P_{\lambda_i}-P_{\lambda_{i-1}}$ is E-finite, therefore $A_\varepsilon^\natural$ is defined, and

$$\|A_\varepsilon^\natural\|\leqq\|A_\varepsilon\|\ \|P^\natural\|,$$

where $\|P^\natural\|$ is a finite number depending only P. Thus we obtain the $A^\natural$ as the limit of $A_\varepsilon^\natural$. Thus defined operation $A^\natural$ satisfies all the conditions required to the $\natural$-operation; this will be clear by the argument of [5].

As usual, we say that an operator $A\in\boldsymbol{M}$ is *finite* if and only if it is contained in some finite projection $P\in\boldsymbol{M}$. Finally we shall extend the $\natural$-operation to any finite operator in $\boldsymbol{M}$. Let a projection $P\in\boldsymbol{M}$ be finite, then as stated in Theorem 1, P is *E-finite* in any $\boldsymbol{M}_{(Z_1)}$, where $\sum_{1\leq i<\infty}\oplus Z_i=I$, therefore we can define the $\natural$-operation in each part $\boldsymbol{M}_{(Z_i)}$ by the above arguments, adding them, we obtain the $P^\natural$. Similarly let $A\in\boldsymbol{M}$ be finite and be contained in a finite projection $P\in\boldsymbol{M}$; let the central decomposition concerning P be Z_i. Then in each Z_i, A is E-finite, therefore we can define the $A^\natural{}_{(Z_i)}$. Adding these $A^\natural{}_{(Z_i)}$, we obtain the $A^\natural$. It may be easily seen that thus obtained $A^\natural$ satisfies the all conditions of the $\natural$-operation. In this place, it is to be remarked that the above $A^\natural$ is not necessarily in $\boldsymbol{M}^\natural$, but may be an unbounded operator. These circumstances will be clarified in the next section.

Summarising the above mentioned and the Dixmier Theorem of the finite class, we obtain our principal.

Theorem 2. *Let $\boldsymbol{M}$ be an arbitrary ring of operators. Then for any finite operator $A\in\boldsymbol{M}$, we can define an operation $A\to A^\natural$ possessing the following properties:*

(1) *If (a finite) $A\in\boldsymbol{M}^\natural$, $A^\natural=A$,*

(2) $(\lambda A)^\natural=\lambda A^\natural$,

(3) *If A and B be finite, then $(A+B)^\natural=A^\natural+B^\natural$,*

(4α) *If AB be finite, then $(AB)^\natural=(BA)^\natural$,*

(4β) *$(AC)^\natural=A^\natural C$ for any $C\in\boldsymbol{M}^\natural$,*

(5α) *If A be self-adjoint and $A\geqq 0$, then $A^\natural$ is also self-adjoint and $A^\natural\geqq 0$,*

(5β) *If A be self-adjoint, $A\geqq 0$ and $A^\natural=0$, then $A=0$,*

(6) $(A^*)^\natural=(A^\natural)^*$.

If A is E-finite , then $A^\natural\in\boldsymbol{M}^\natural$.

Furthermore, if there exists another finite projection F such that its central envelope $Z'=I$, then we can define another $\natural$-operation $\natural'$, say; then these operations are related by

4) See [3], Footnote 43) of p. 391.

$$A^{\natural\prime}=(F^{\natural P})^{-1}E^{\natural P}A^{\natural} \quad \textit{for any finite } A\in \boldsymbol{M},$$

where P denotes the least projection containing E and F.

The latter half is due to [5; Theorem 3].

3. First we consider only the *abelian* ring of operators, and denote it simply by $\boldsymbol{M}$.

As be well-known, $\boldsymbol{M}$ is an abelian B^*-algebra, therefore we can represent $\boldsymbol{M}$ as the set of all continuous functions on the compact Hausdorff space Ω[5]. But this Ω is also considered as the Boolean space of the complete Boolean algebra $\boldsymbol{P}$ generated by the projections of $\boldsymbol{M}$, and there exists a one-to-one correspondence between the Boolean algebra $\boldsymbol{\Gamma}$ generated by the open and closed sets on Ω and the $\boldsymbol{P}$. If we consider this Boolean algebra $\boldsymbol{\Gamma}$ as a σ-Boolean algebra, then we can define the measurability of the functions on Ω with respect to this $\boldsymbol{\Gamma}$[6]. That is, a function $f(\xi)$ on Ω is called *measurable* if and only if the set $\{\xi\in\Omega;\ f(\xi)<c\}\in\boldsymbol{\Gamma}$, for every real number c. Moreover, if we denote the set $\{\xi\in\Omega;\ n-1\leqq|f(\xi)|<n\}$ by $\boldsymbol{\Gamma}_n$, then $f(\xi)$ is called *almost everywhere measurable* if and only if, $\Gamma_n\in\boldsymbol{\Gamma}$ $f(\xi)$ is measurable on Γ_n for every integer n, and $\Omega=\sum_{1\leq n<\infty}\oplus\Gamma_n$. Where $\sum\oplus\Gamma_n$ denotes the least open and closed set containing all Γ_n.

Then we have the following characterisation of the abelian rings of operators, which is a generalisation of [3; Theorem 6].

Theorem 3. *Let $\boldsymbol{M}$ be an abelian ring of operators. Then $\boldsymbol{M}$ is characterised by the set of all bounded measurable functions on the Boolean space Ω.*

Proof. Let $f(\xi)$ be a bounded measurable functions on Ω with bound c, and let the set $\{\xi\in\Omega;\ f(\xi)<\lambda_n\}$ be $\Gamma_{\lambda n}\in\boldsymbol{\Gamma}$, then there corresponds a projection $E(\lambda_n)\in\boldsymbol{M}$ for Γ_{λ_n}. Consider $\sum_{n=1}^{m}\lambda_n<(E(\lambda_n)-E(\lambda_{n-1}))\ x,y>$ for $x,y\in\boldsymbol{H}$, then it is easily seen that $|\sum_{n=1}^{m}\lambda_n<(E(\lambda_n)-E(\lambda_{n-1}))\ x,y>|\leqq c.\|x\|.\|y\|$. Therefore, by the well-known process, we obtain a Lebesgue-Stieltjes integral $\int\lambda d<E(\lambda)\,x,y>$ as the limit $m\to\infty$, for any $x,y\in\boldsymbol{H}$, and $|\int\lambda d<E(\lambda)x,y>|\leqq c.\|x\|.\|y\|$. Thus there exists a bounded self-adjoint operator A such that

$$(4) \qquad <Ax,y>=\int\lambda d<E(\lambda)\,x,y>$$ [7]

As $\boldsymbol{M}$ is weakly closed and $E(\lambda)\in\boldsymbol{M}$, we have $A\in\boldsymbol{M}$.

The converse is evident by the spectral decomposition and [2; Theorem 1].

---

5) See R. Arens, On a theorem of Gelfand and Neumark, Proc. Nat. Acad. Sci., 32 (1946), 237-239.

6) Concerning to this notion of the measurability, see P. R. Halmos, Measure Theory (1950).

7) See [6; Chap. VII].

Lemma 3. 1. *Let $f(\xi)$ be an a.e. measurable function on Ω, then there exists a self-adjoint operator A with the dense domain such that*

$$(5) \qquad <Ax, y> = \int \lambda d <E(\lambda)x, y>$$

for $x \in \boldsymbol{H}$, in which $\int |\lambda|^2 d<E(\lambda)x, y>$ is defined, and for any $y \in \boldsymbol{H}$. Moreover, this x is characterised by the fact that x is in the domain of A.

Proof. This follows easily by the above arguments and by [6; Chap. VII, § 2], so that we omit the proof.

Consider again the general ring of operators $\boldsymbol{M}$. In the terminology of this section Theorem 1 can be stated as follows: *A projection $P \in \boldsymbol{M}$ is finite if and only if, P is a. e. E-finite in the central decomposition* (2). This statement may be justified by the next theorem.

Theorem 4. *Let an operator $A \in \boldsymbol{M}$ be finite, and let $A^{\natural}$ be the image of A by the $\natural$-operation defined above, then $A^{\natural}$ can be represented by the a. e. measurable function on the Boolean space Ω, corresponding to the Boolean algebra $\boldsymbol{P}$ of the projection in $\boldsymbol{M}^{\natural}$. A is E-finite if and only if it is a bounded function on Ω.*

Proof. Evident by Theorem 3, Lemma 3.1, and by the definition of the $\natural$-operation.

Now denote by $\overline{\boldsymbol{M}}$ the set of all operators $A \ \eta \ \boldsymbol{M}$[8], that is, $UAU^{-1} = A$ for all unitary operators $U \in \boldsymbol{M}'$. Then the above theorem can be expressed in the following way:

Theorem 5. *Let an operator $A \in \boldsymbol{M}$ be finite, then $A^{\natural} = \overline{\boldsymbol{M}}^{\natural}$, and $A^{\natural} \in \boldsymbol{M}^{\natural}$ if and only if A is E-finite.*

Proof. It is well-known that A is commutative to a bounded self-adjoint operator B if and only if every resolution of the identity $E(\lambda)$ is commutative to B[9]. Because the expression (4) or (5) can be considered as the spectral decomposition of $A^{\natural}$, we obtain the theorem.

References.

[1] J. Dixmier: Les anneaux d'opérateurs de classe finie. Ann. Ecole Norm. Sup. 66 (1949), 209-261.

[2] J. von Neumann: Zur Algebra der Funktionaloperatoren und Theorie der normalen Operatoren, Math. Ann., 102 (1929), 370-427.

[3] ———: Über Funktionen von Funktionaloperatoren, Ann. of Math., 32 (1931), 191-226.

[4] F. J. Murray and J. von Neumann: On rings of operators, Ann. of Math., 37 (1936), 116-229.

[5] H. Sunouchi: On rings of operators of infinite classes. Proc. Japan Acad., 28 (1952), 9-13.

[6] B. Sz. Nagy: Spektraldarstellung linearen Transformationen des Hilbertschen Raumes, Ergebnisse d. Math. 5. 2 (1942).

8) For this notion, see [4; Def. 4.2.1.].

9) See for example, [6], p. 50.