10. *Probability-theoretic Investigations on Inheritance. XVI$_3$. Further Discussions on Interchange of Infants*

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6. An intermediate problem

The discussions in the previous sections have based upon a pair consisting of mother and an apparent child as the unit of consideration, while those in the preceding chapter concerned a triple consisting of parents and an apparent child. We shall now discuss a problem of detecting the interchange of infants which is situated in an intermediate position.

Let now *a triple consisting of a child and its parents and a pair consisting of a child and its mother* be given under a suspicion of interchange of infants. We then consider the probability of an event that the decision is possible under a supposition of actual interchange; cf. the remark stated at the end of §1 and also at the beginning of §6 in XV. The basic tools of attack on the present problem have been made ready.

In conformity to (5.2) of XV, let us designate by $G_0(ij, hk)$ the probability of an event that the detection of interchange is possible within a triple alone which consists of a mother A_{ij}, a father A_{hk} and an apparent child. Since now a mother-child combination is presented instead of a mating-child combination, the second quantity in (5.2) of XV is here to be replaced by the quantity

$$\Psi_*(ij,\ hk) \tag{6.1}$$

representing the *probability of an event that the detection becomes possible only by taking the mother-child combination into account.* The probability of an event that such a triple is presented and the detection is possible against a pair consisting of a mother and an apparent child, is thus given by the sum

$$\mathfrak{G}(ij, hk)=G_0(ij, hk)+\Psi_*(ij, hk). \tag{6.2}$$

Concerning the first term of the second member in (6.2), we have discoursed fully in the preceding chapter. The second term $\Psi_*(ij, hk)$ possesses an analogous structure as $\Phi(ij, hk)$. In fact, according to the present situation, we have only to replace the φ's contained in the latter by the corresponding ψ's. We thus obtain the following expressions:

(6.3) $\Psi_*(ii, ii)=0,$

(6.4) $\Psi_*(ii, hh)=0 \qquad (h\neq i),$

(6.5) $$\Psi_*(ii, ih)=\overline{A}_{ii}\overline{A}_{ih}\{\tfrac{1}{2}\psi(-ii, +ih)+\tfrac{1}{2}\psi(-ih, +ii)\} = p_i^4p_h^2(1-p_i) \qquad (h\neq i),$$

(6.6) $$\Psi_*(ii, hk)=\overline{A}_{ii}\overline{A}_{hk}\{\tfrac{1}{2}\psi(-ih, +ik)+\tfrac{1}{2}\psi(-ik, +ih)\} = p_i^3p_hp_k((1-p_i)(p_h+p_k)-2p_hp_k) \qquad (h, k\neq i;\ h\neq k);$$

(6.7) $\Psi_*(ij, ii)=\Psi_*(ii, ij)=p_i^4p_j^2(1-p_i) \qquad (i\neq j),$

(6.8) $$\Psi_*(ij, ij)=\overline{A}_{ij}^2\{\tfrac{1}{4}\psi(-ii, +jj+ij)+\tfrac{1}{4}\psi(-jj, +ii+ij) +\tfrac{1}{2}\psi(-ij, +ii+jj)\} = p_i^2p_j^2(p_i^2+p_j^2+2p_ip_j-2p_ip_j(p_i+p_j)) \qquad (i\neq j),$$

(6.9) $$\Psi_*(ij, hh)=\Psi_*(hh, ij)=p_ip_jp_h^3(p_i+p_j-2p_ip_j-(p_i+p_j)p_h) \qquad (i\neq j;\ h\neq i, j),$$

(6.10) $$\begin{aligned}\Psi_*(ij, ih)=\overline{A}_{ij}\overline{A}_{ih}\{&\tfrac{1}{4}\psi(-ii, +ij+ih+jh)+\tfrac{1}{4}\psi(-ij, +ii+ih+jh)\\ &+\tfrac{1}{4}\psi(-ih, +ii+ij+jh)+\tfrac{1}{4}\psi(-jh, +ii+ij+ih)\}\\ =p_i^2p_jp_h(&2p_ip_j(1-p_i)+(p_i+p_j)p_i(1-p_j)\\ &+(3p_i+4p_j-3p_i^2-p_j^2-8p_ip_j)p_h-(p_i+p_j)p_h^2)\\ &(i\neq j;\ h\neq i, j),\end{aligned}$$

(6.11) $$\begin{aligned}\Psi_*(ij, hk)=\overline{A}_{ij}\overline{A}_{hk}\{&\tfrac{1}{4}\psi(-ih, +jh+ik+jk)+\tfrac{1}{4}\psi(-jh, +ih+ik+jk)\\ &+\tfrac{1}{4}\psi(-ik, +ih+jh+jk)+\tfrac{1}{4}\psi(-jk, +ih+jh+ik)\}\\ =p_ip_jp_hp_k(&(4(p_i+p_j)-(p_i^2+p_j^2)-6p_ip_j)(p_h+p_k)\\ &-(p_i+p_j)(p_h^2+p_k^2)-6(p_i+p_j)p_hp_k)\\ &(i\neq j;\ h, k\neq i, j;\ h\neq k).\end{aligned}$$

All the possible cases have thus essentially been exhausted. By summing up the partial probabilities in (6.3) to (6.11) over all possible types of father for a fixed type of mother, we get

(6.12) $$\begin{aligned}\Psi_*(ii)&\equiv\Psi_*(ii, ii)+\sum_{h\neq i}(\Psi_*(ii, hh)+\Psi_*(ii, ih))+\sum_{h, k\neq i}{}'\Psi_*(ii, hk)\\ &=p_i^3(S_2-S_3-S_2^2+S_4-(S_2-S_3)p_i-(1-2S_2)p_i^2+2p_i^3-3p_i^4),\end{aligned}$$

(6.13) $$\begin{aligned}\Psi_*(ij)\equiv&\Psi_*(ij, ii)+\Psi_*(ij, jj)+\Psi_*(ij, ij)\\ &+\sum_{h\neq i, j}(\Psi_*(ij, ih)+\Psi_*(ij, jh)+\Psi_*(ij, hh))+\sum_{h, k\neq i, j}{}'\Psi_*(ij, hk)\\ =&p_ip_j((4S_2-4S_3-3S_2^2+3S_4)(p_i+p_j)-(2S_2-S_3)(p_i^2+p_j^2)\\ &-2(3S_2-2S_3)p_ip_j-(3-4S_2)(p_i^3+p_j^3)\\ &-(1-4S_2)p_ip_j(p_i+p_j)+5(p_i^4+p_j^4)+5p_ip_j(p_i^2+p_j^2)-2p_i^2p_j^2\\ &-5(p_i^5+p_j^5)-6p_ip_j(p_i^3+p_j^3)) \qquad (i\neq j).\end{aligned}$$

The sums of (5.27) of XV and (6.12), and of (5.28) of XV and (6.13) then become

(6.14) $$\begin{aligned}\mathfrak{G}(ii)\equiv G_0(ii)+\Psi_*(ii)=p_i^2(&1-(3S_2-S_3+S_2^2-S_4)p_i\\ &-(S_2-S_3)p_i^2+(1+2S_2)p_i^3+p_i^4-3p_i^5),\end{aligned}$$

$$
\begin{aligned}
(6.15)\quad \mathfrak{G}(ij) \equiv G_0(ij) + \Psi_*(ij) = p_i p_j (2 &- (4S_2 + 3S_2^2 - 3S_4)(p_i + p_j) \\
&- (2S_2 - S_3)(p_i^2 + p_j^2) - 2(3S_2 - 2S_3)p_i p_j + (1 + 4S_2)(p_i^3 + p_j^3) \\
&- (1 - 4S_2)p_i p_j (p_i + p_j) + 3(p_i^4 + p_j^4) + 5p_i p_j (p_i^2 + p_j^2) + 6p_i^2 p_j^2 \\
&- 5(p_i^5 + p_j^5) - 6p_i p_j (p_i^3 + p_j^3)) \qquad (i \neq j),
\end{aligned}
$$

respectively. On the other hand, we get by summation

$$
(6.16)\quad \sum_{i=1}^{m} \Psi_*(ii) = S_2S_3 - S_5 - S_3^2 - S_2S_4 + 2S_6 - S_2^2S_3 + 2S_3S_4 + 2S_2S_5 - 3S_7,
$$

$$
\begin{aligned}
(6.17)\quad \sum_{i,j}{}' \Psi_*(ij) = 4S_2^2 - 3S_4 - 11S_2S_3 + 9S_5 - 6S_2^3 + 4S_3^2 + 17S_2S_4 - 14S_6 \\
+ 9S_2^2S_3 - 6S_3S_4 - 14S_2S_5 + 11S_7.
\end{aligned}
$$

The sums of (5.33) of XV and (6.16), and of (5.34) of XV and (6.17) become

$$
(6.18)\quad \sum_{i=1}^{m} \mathfrak{G}(ii) = S_2 - 3S_2S_3 + S_5 + S_3^2 - S_2S_4 + S_6 - S_2^2S_3 + 2S_3S_4 + 2S_2S_5 - 3S_7,
$$

$$
\begin{aligned}
(6.19)\quad \sum_{i,j}{}' \mathfrak{G}(ij) = 1 - S_2 - 4S_2^2 + S_4 + S_2S_3 + 3S_5 - 6S_2^3 + 4S_3^2 + 17S_2S_4 - 16S_6 \\
+ 9S_2^2S_3 - 6S_3S_4 - 14S_2S_5 + 11S_7.
\end{aligned}
$$

On the other hand, the sum of (6.16) and (6.17) becomes

$$
\begin{aligned}
(6.20)\quad \Psi_* = 4S_2^2 - 3S_4 - 10S_2S_3 + 8S_5 - 6S_2^3 + 3S_3^2 + 16S_2S_4 - 12S_6 \\
+ 8S_2^2S_3 - 4S_3S_4 - 12S_2S_5 + 8S_7.
\end{aligned}
$$

Finaly, the sum of (5.39) of XV and (6.20) or, which is the same thing, the sum of (6.18) and (6.19) yields the *whole probability of detecting the interchange of infants*:

$$
\begin{aligned}
(6.21)\quad \mathfrak{G} = G_0 + \Psi_* = 1 - 4S_2^2 + S_4 - 2S_2S_3 + 4S_5 - 6S_2^3 + 5S_3^2 + 16S_2S_4 - 15S_6 \\
+ 8S_2^2S_3 - 4S_3S_4 - 12S_2S_5 + 8S_7.
\end{aligned}
$$

7. An alternative procedure

The same result on the whole probability as stated in (6.21) can be obtained by an alternative procedure. Namely, in conformity to (2.1), let us designate by $F_0(ij)$ the probability of an event that the detection of interchange is possible within a pair alone which consists of a mother A_{ij} and an apparent child. Since now a mating-child combination is presented instead of a mother-child combination, the second quantity in (2.1) is here to be replaced by the quantity

$$
(7.1)\quad \Phi_*(ij)
$$

representing the probability of an event that the detection becomes possible only by taking the mating-combination into account. The probability of an event that such a pair is presented and the detection is possible against a triple consisting of a mating and an apparent child, is thus given by the sum

$$
(7.2)\quad \mathfrak{F}(ij) = F_0(ij) + \Phi_*(ij).
$$

Concerning the first term of the second member in (7.2), we have discoursed fully in the preceding section. The second term $\Phi_*(ij)$ possesses an analogous structure as $\Psi(ij)$. In fact, according to the present situation, we have only to replace the ψ's contained in the latter by the corresponding φ's. We thus obtain

$$(7.3)\quad \begin{aligned}\Phi_*(ii) &= \bar{A}_{ii}\{p_i\varphi(-ii, +\sum_{h\neq i} ih) + \sum_{h\neq i} p_h\varphi(-ih, +ii+\sum_{k\neq i,h} ik)\} \\ &= p_i^3(2(1-2S_2+S_3)-(1+2S_2-3S_3)p_i+(1+2S_2)p_i^2 \\ &\qquad +2p_i^3-5p_i^4),\end{aligned}$$

$$(7.4)\quad \begin{aligned}\Phi_*(ij) = \bar{A}_{ij}\{&\tfrac{1}{2}p_i\varphi(-ii, +jj+ij+\sum_{h\neq i,j}(ih+jh)) \\ &+\tfrac{1}{2}p_j\varphi(-jj, +ii+ij+\sum_{h\neq i,j}(ih+jh)) \\ &+\tfrac{1}{2}(p_i+p_j)\varphi(-ij, +ii+jj+\sum_{h\neq i,j}(ih+jh)) \\ &+\sum_{h\neq i,j}\tfrac{1}{2}p_h\varphi(-ih, +ii+jj+ij+\sum_{k\neq i,j,h} ik+\sum_{k\neq i,j} jk) \\ &+\sum_{h\neq i,j}\tfrac{1}{2}p_h\varphi(-jh, +ii+jj+ij+\sum_{k\neq i,j} ik+\sum_{k\neq i,j,h} jk)\} \\ = p_ip_j(&2(2-2S_2+S_3)(p_i+p_j)-(2+2S_2-3S_3)(p_i^2+p_j^2) \\ &-4(1+3S_2-S_3)p_ip_j+(1+2S_2)(p_i^3+p_j^3) \\ &-2(1-3S_2)p_ip_j(p_i+p_j)+2(p_i^4+p_j^4)+8p_ip_j(p_i^2+p_j^2)+4p_i^2p_j^2 \\ &-5(p_i^5+p_j^5)-6p_ip_j(p_i^3+p_j^3)-2p_i^2p_j^2(p_i+p_j)) \qquad (i\neq j).\end{aligned}$$

The sums of (2.3) and (7.3), and of (2.4) and (7.4) become

$$(7.5)\quad \mathfrak{F}(ii) = p_i^2(1-2(2S_2-S_3)p_i-(2S_2-3S_3)p_i^2+(1+2S_2)p_i^3+2p_i^4-5p_i^5),$$

$$(7.6)\quad \begin{aligned}\mathfrak{F}(ij) = p_ip_j(&2-2(2S_2-S_3)(p_i+p_j)-(2S_2-3S_3)(p_i^2+p_j^2) \\ &-(3S_2-S_3)p_ip_j+(1+2S_2)(p_i^3+p_j^3)-2(1-3S_2)p_ip_j(p_i+p_j) \\ &+2(p_i^4+p_j^4)+8p_ip_j(p_i^2+p_j^2)+4p_i^2p_j^2-5(p_i^5+p_j^5)-6p_ip_j(p_i^3+p_j^3) \\ &-2p_i^2p_j^2(p_i+p_j)) \qquad (i\neq j).\end{aligned}$$

Further, summing up the probabilities (7.3) and (7.4) over respective possible suffices, we obtain

$$(7.7)\quad \begin{aligned}\sum_{i=1}^{m}\Phi_*(ii) &= 2S_3-S_4-4S_2S_3+S_5+2S_3^2-2S_2S_4+2S_6+3S_3S_4 \\ &\qquad +2S_2S_5-5S_7,\end{aligned}$$

$$(7.8)\quad \begin{aligned}\sum_{i,j}{}'\Phi_*(ij) &= 4S_2-6S_3-6S_2^2+5S_4+2S_2S_3+3S_5-6S_2^3+3S_3^2 \\ &\qquad +18S_2S_4-17S_6+8S_2^2S_3-7S_3S_4-14S_2S_5+13S_7.\end{aligned}$$

Further summations yield

$$(7.9)\quad \sum_{i=1}^{m}\mathfrak{F}(ii) = S_2-4S_2S_3+S_5+2S_3^2-2S_2S_4+2S_6+3S_3S_4+2S_2S_5-5S_7,$$

$$(7.10)\quad \begin{aligned}\sum_{i,j}{}'\mathfrak{F}(ij) &= 1-S_2-4S_2^2+S_4+2S_2S_3+3S_5-6S_2^3+3S_3^2+18S_2S_4 \\ &\qquad -17S_6+8S_2^2S_3-7S_3S_4-14S_2S_5+13S_7;\end{aligned}$$

$$(7.11)\quad \begin{aligned}\Phi_* \equiv \sum_{i\leq j}\Phi_*(ij) &= 4S_2-4S_3-6S_2^2+4S_4-2S_2S_3+4S_5 \\ &\qquad -6S_2^3+5S_3^2+16S_2S_4-15S_6+8S_2^2S_3-4S_3S_4-12S_2S_5+8S_7.\end{aligned}$$

The sum of (7.9) and (7.10) or also of (2.15) and (7.11) yields the *whole probability of detecting the interchange*:

$$
\begin{aligned}
\mathfrak{F} &= F_0 + \Phi_* \\
\text{(7.12)} \quad &= 1 - 4S_2^2 + S_4 - 2S_2S_3 + 4S_5 - 6S_2^3 + 5S_3^2 + 16S_2S_4 - 15S_6 \\
&\quad + 8S_2^2S_3 - 4S_3S_4 - 12S_2S_5 + 8S_7.
\end{aligned}
$$

The final result (7.12) coincides, of course, with the previous one, namely, $\mathfrak{G}$ obtained in (6.21).

Correction

A correction should be made for the expression (2.6) (these Proc. **25** (1952), p. 541), since it contains a mistake in claculation. It should be read:

$$
\begin{aligned}
\Psi(ij) &= \overline{A}_{ij}\{\tfrac{1}{2}p_i\psi(-ii,\ +jj+ij+\sum_{h\neq i,j}(ih+jh)) \\
&\quad + \tfrac{1}{2}p_j\psi(-jj,\ +ii+ij+\sum_{h\neq i,j}(ih+jh)) \\
&\quad + \tfrac{1}{2}(p_i+p_j)\psi(-ij,\ +ii+jj+\sum_{h\neq i,j}(ih+jh)) \\
\text{(2.6)} \quad &\quad + \sum_{h\neq i,j}\tfrac{1}{2}p_h\psi(-ih,\ +ii+jj+ij+\sum_{k\neq i,j,h}ik+\sum_{k\neq i,j}jk) \\
&\quad + \sum_{h\neq i,j}\tfrac{1}{2}p_h\psi(-jh,\ +ii+jj+ij+\sum_{k\neq i,j}ik+\sum_{k\neq i,j,h}jk)\} \\
&= p_ip_j((3-5S_2+2S_3)(p_i+p_j)-(4-3S_2)(p_i^2+p_j^2) \\
&\quad -2(4-3S_2)p_ip_j+5(p_i^3+p_j^3)+8p_ip_j(p_i+p_j) \\
&\quad -4(p_i^4+p_j^4)-6p_ip_j(p_i^2+p_j^2)-4p_i^2p_j^2) \qquad (i\neq j).
\end{aligned}
$$

Accordingly, the subsequent expressions should be corrected as follows:

$$
\begin{aligned}
F(ij) &= p_ip_j(2-(1+5S_2-2S_3)(p_i+p_j)-(2-3S_2)(p_i^2+p_j^2) \\
\text{(2.8)} \quad &\quad -2(2-3S_2)p_ip_j+5(p_i^3+p_j^3)+8p_ip_j(p_i+p_j) \\
&\quad -4(p_i^4+p_j^4)-6p_ip_j(p_i^2+p_j^2)-4p_i^2p_j^2) \qquad (i\neq j).
\end{aligned}
$$

$$
\begin{aligned}
\text{(2.12)} \quad \sum_{i,j}{}' \Psi(ij) &= 3S_2 - 7S_3 - 9S_2^2 + 13S_4 \\
&\quad + 18S_2S_3 - 17S_5 + 3S_2^3 - 4S_3^2 - 12S_2S_4 + 12S_6.
\end{aligned}
$$

$$
\begin{aligned}
\text{(2.14)} \quad \sum_{i,j}{}' F(ij) &= 1 - 2S_2 - S_3 - 7S_2^2 + 9S_4 \\
&\quad + 18S_2S_3 - 17S_5 + 3S_2^3 - 4S_3^2 - 12S_2S_4 + 12S_6;
\end{aligned}
$$

$$
\begin{aligned}
\text{(2.16)} \quad \Psi \equiv \sum_{i\leq j} \Psi(ij) &= 3S_2 - 6S_3 - 9S_2^2 + 11S_4 \\
&\quad + 16S_2S_3 - 14S_5 + 3S_2^3 - 3S_3^2 - 10S_2S_4 + 9S_6.
\end{aligned}
$$

$$
\begin{aligned}
F &= F_0 + \Psi \\
\text{(2.17)} \quad &= 1 - S_2 - 2S_3 - 7S_2^2 + 8S_4 \\
&\quad + 16S_2S_3 - 14S_5 + 3S_2^3 - 3S_3^2 - 10S_2S_4 + 9S_6.
\end{aligned}
$$

The inequalities (3.5) and (3.6) (p. 543) remain valid.

However, the expression (5.4) (p. 546) and hence the subsequent expression for its derivative should be corrected as follows:

$$
(F)^{\text{stat}} = \left(1-\frac{1}{m}\right)\left(1-\frac{9}{m^2}+\frac{18}{m^3}-\frac{9}{m^4}\right)
$$

$$(5.4) \quad \frac{d}{d(1/m)}(F)^{\text{stat}} = -\left(1-\frac{2}{m}\right)\left(\left(1-\frac{2}{m}\right)\left(1+\frac{22}{m}\right)+\frac{3}{m^2}+\frac{26}{m^3}\right) - \frac{7}{m^4} < 0 \qquad (m \geqq 2).$$

By the way, some other misprints should be pointed out: the right-hand members of the second and the third expressions (7.13) (p. 535) are to be read $v_1 v_2^2(v+v_2)u(1+v)$ and $v_2^4(u(1+v)+v_1(v+v_2))$, instead of $v_1 v_2(v+v_2)u(1+v)$ and $v_2^4(u(1+v)+2uv_1(1+v)(v+v_2))$, respectively.

—*To be continued*—