4. On the Structure of the Plane Translation of Brouwer

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The so-called plane translation theorem of Brouwer[1] had been a starting point to a series of investigations concerning the sense preserving topological transformation of the Euclidean plane onto itself without fixed point[2][3][4][5][6][7]. But the general behavior of such a transformation, called by Scorza Dragoni[5] a generalized translation, is not so simple as one would expect at first sight and but little is definitely known until now. In this note we shall communicate without proof the result of our investigation which purports to determine the complete structure of the generalized translation, given in the main theorem at the end of this note[*].

1. Throughout this note f denotes a sense preserving topological transformation of the Euclidean plane E^2 onto itself without fixed point. The n-th iteration of f will be denoted by f^n for any integer n.

2. If M is a point set, the cluster set of $f^n(M)$ for all positive integer n will be called the (+)-*limit* of M and denoted by $\lim^+ M$; likewise for (−)-*limit*.

If $\lim^+ M$ is non vacuous, then M is said to be (+)-*irregular*. A point p is said to be (+)-*irregular*, if every neighbourhood $U(p)$ of p, where we understand by a neighbourhood always a domain containing the point in question, is (+)-irregular. If furthermore $P=\bigcup \lim^+ U(p)$ for all neighbourhoods $U(p)$ of p is non vacuous, then p is said to be *strongly* (+)-*irregular*. If P vanishes, then p is said to be *weakly* (+)-*irregular*.

Similarly for (−)-*irregularity*[**].

If a point p is neither (+)-nor (−)-irregular, then p is said to be *regular*. We can construct a simple *example of f for which all points of the plane are both* (+)- *and* (−)-*irregular*.

$P=\bigcup \lim^+ U(p)$ will be called the (+)-*singularity polar to* p, and p a *pole* of P. Similarly for (−)-*singularity*[***].

Duality Theorem. If p is strongly (+)-*irregular, then every point q of the* (+)-*singularity P polar to p are strongly* (−)-*irregu-*

[*] Full account will appear in Osaka Math. Jour., **5**.

[**] Since a proposition remains true if we interchange (+) and (−), we omit in the sequel the propositions thus obtained.

[***] The notion of singularity has already been mentioned by Sperner[6].

lar, and the $(-)$*-singularity polar to* q *passes through* p.

Lemma 1. If U *is a* $(+)$*-irregular domain, then* $\overline{U}$ *contains at least a strongly* $(+)$*-irregular point.*

3. If α is a *translation arc*, that is an arc which has with its image an end point in common, the interior of the arc being free, then $\bigcup_{-\infty}^{\infty} f^n(\alpha)$ will be called a *stream-line* (Bahnkurve according to Brouwer). If a stream-line is an open line, i.e., a closed set which is a topological image of a straight line, it will be called *regular*. If every stream-line through a point p is not regular, p will be said to be *irregular in the large.*

Remark. A regular point can be irregular in the large.

4. A simply connected domain which is bounded by two open lines, one of which is the image of the other, is called a *translation field* (Brouwer[1]). It is the important so-called "plane translation theorem of Brouwer" that any point is contained in some translation field.

If T is a translation field, $A=\bigcup_{-\infty}^{\infty} f^n(\overline{T})$ will be called the *area of translation* generated by T. A is simply connected and bounded by $\lim^+T=B^+(A)$ and $\lim^-T=B^-(A)$.

Let b be an accessible point of $B^+(A)$ and let bc be a free arc wholly contained in A except for b. Then c can be joined to $f(c)=c'$ in A by a translation arc cc' such that $bc+cc'+f(bc)$ forms an arc, called a *link*, which has only $f(bc)$ in common with its image. The link bounds with a part of $B^+(A)$ a domain C, which we shall call a *bordering cell*. If B_b denotes the subset of $B^+(A)$ which makes part in the boundary of the bordering cell, $B=\bigcup_{-\infty}^{\infty} f^n(B_b)$ will be called a *boundary block*.

A boundary block is either an unbounded connected set or composed of unbounded continua and is invariant under f. $\lim^+B_b$ and $\lim^-B_b$ will be called the $(+)$- and the $(-)$-*end* of the boundary block; the ends of a boundary block consist likewise of unbounded continua, each point of which is inaccessible from A. The ends are again split into blocks, each one of which is invariant under f.

If p is a strongly $(+)$-irregular point, the $(+)$-singularity P polar to p is contained in some $B^+(A)$ and each boundary block of $B^+(A)$ is either disjoint from or wholly contained in P. If a boundary block B is wholly contained in P, the closure S of B will be called a (*strong*) $(+)$*-singularity block*, where B is the *interior* of S and the ends of B are the *ends* of S. The component of E^2-S which contains A is called the *positive side* of S. A $(+)$-singularity block moves either in the positive or in the negative direction along itself under f. Similarly for $(-)$-singularity blocks.

The importance of singularity blocks is that these are uniquely determined by f, whereas stream-lines admit freedom of choice and does not necessarily represent the peculiarities of f in the neighbourhood of each point (cf. Remark in § 3).

Lemma 2. Two (+)-singularity blocks whose positive sides have points in common cannot intersect. They have some interior points in common, if and only if the (−)-end of one of them is non vacuous.

Lemma 3. The (+)-end of a (+)-singularity block cannot have a common part with the (−)-end of another (+)-singularity block, if their positive sides have points in common.

Lemma 4. A (+)-singularity block cannot intersect a (−)-singularity block if its (+)-end vanishes.

Lemma 5. If S_1, S_2, ... are a sequence of singularity blocks such that for each S_i all the other S_n belong to the same component of E^2-S_i, then they have no cluster set.

If the condition of this lemma is not fulfilled, singularity blocks may have cluster set. Indeed, through every weakly (+)-irregular point passes the cluster set of a certain sequence of (+)-singularity blocks. Such a cluster set will be called a *weak (+)-singularity block.*

5. The set of all regular points makes evidently an open set. Each component of this set will be called a *maximal regular domain.* A maximal regular domain may be bounded and/or may be free. If it is not free, it coincides with its images, and will be called an *area of total regularity.* The theorem of Kerékjártó—Sperner[2)3)6)], yields at once:

An area of total regularity can be filled with a regular family of regular stream-lines.

6. Summing up we obtain the following main theorem:

Structure Theorem. Let f be a generalized translation, i. e., a sense preserving topological transformation of E^2 onto itself without fixed point. Then the plane E^2 is divided into three kinds of disjoint sets: O_1, O_2,...; O_1', O_2',...; and F. Each O_n, the area of total regularity, is a simply connected domain and can be filled with a regular family of regular stream-lines. Each O_n' is a simply connected free domain and its points are all regular. F is closed, consists of all irregular points of f and is filled with disjoint strong and weak singularity blocks which are uniquely determined by f.

Conversely, a generalized translation can be constructed by taking certain sets having the structure of singularity blocks and fulfilling the conditions of lemmas 2, 3, 4, 5 and some others as singularity blocks.

References

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