2. On Homotopy Classification and Extension

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In the present note we give a homotopy classification theorem for mappings of an $(n+k)$-dimensional complex into a finite complex Y such that

(1) $\pi_i(Y)=0$ for $0\leqq i<n$ and $n<i<n+k$,

and the corresponding extension theorem, where $n\geqq k+2$ and $k\leqq 6$.[1)2)]

§1. Let $X(\ni x_0)$ be an arcwise connected and simply connected space, and let $X^*=X\cup e^{n_1}\cup\cdots\cup e^{n_s}$, where the boundary $\dot{e}^{n_i}(\ni q_i)$ of each cell e^{n_i} is attached to X by a map $f_i:(\dot{e}^{n_i},q_i)\to(X,x_0)$. We refer to such a space X^* as $\{X|e^{n_1},\cdots,e^{n_s};f_1,\cdots,f_s\}$. Suppose that $g_i:(E^r,\dot{E}^r,p_0)\to(e^{n_i},\dot{e}^{n_i},q_i)$ $(i=1,\cdots,t\leqq s)$ is a representative of the homotopy class $\{g_i\}\in\pi_r(e^{n_i},\dot{e}^{n_i},q_i)$, and that a condition $\sum_{i=1}^{t}\{f_i\circ(g_i|\dot{E}^r)\}=0$ is satisfied in $\pi_{r-1}(X,x_0)$. Then we can construct a map of an r-sphere S^r in X^* as follows: Let ε_i^r $(i=1,2,\cdots,t)$ be t disjoint r-cells in S^r which have a single point p in common, and let $\varepsilon^r=\bigcup_{i=1}^{t}\varepsilon_i^r$, $\dot{\varepsilon}^r=\bigcup_{i=1}^{t}\dot{\varepsilon}_i^r$. Choose an orientation of S^r, and orient each ε_i^r in agreement with S^r. If we map each ε_i^r to e^{n_i} by the map g_i, we get a map $g':(\varepsilon^r,\dot{\varepsilon}^r,p)\to(X^*,X,x_0)$ such that $g'|\dot{\varepsilon}^r$ is null-homotopic in X. Map now $\overline{S^r-\varepsilon^r}$ in X by an arbitrary null-homotopy of $g'|\dot{\varepsilon}^r$, then we obtain a map of S^r in X^*. This is the desired map and such a map is denoted by $<g_1,g_2,\cdots,g_t|X>$.

As for the spherical-maps, we use the following notations: $i_r:S^r\to S^r$ $(r\geqq 1)$ is the identity map; $\eta_r:S^{r+1}\to S^r$ $(r\geqq 2)$, $\nu_r:S^{r+3}\to S^r$ $(r\geqq 4)$ are the suspensions of the Hopf maps η_2, ν_4 respectively. Let $\partial_n:\pi_{n+1}(e^{r+1},\dot{e}^{r+1})\approx\pi_n(S^r)$ be the homotopy boundary, then we refer to maps in the homotopy classes $\partial_r^{-1}\{i_r\}$, $\partial_{r+1}^{-1}\{\eta_r\}$, $\partial_{r+3}^{-1}\{\nu_r\}$ as $\bar{i}_{r+1}$, $\bar{\eta}_{r+1}$, $\bar{\nu}_{r+1}$ respectively.

§2. Using the homology theory of Abelian groups due to Eilenberg—MacLane[3)] and the known results relative to the homotopy

1) Full details will appear in the Journal of the Institute of Polytechnics, Osaka City University. The first part of the details was already presented to the editor of the journal.

2) A general theory of this problem was given by S. Eilenberg—S. MacLane (cf. Proc. Nat. Acad. Sci., U.S.A., IV).

3) S. Eilenberg—S. MacLane: Cohomology theory of Abelian groups and homotopy theory II. Proc. Nat. Acad. Sci., U.S.A., **36**, No. 11 (1950); IV ibid., **38**, No. 4 (1952).

groups of spheres[4)], we can construct a concrete $(n+k)$-dimensional cell complex $R^k_n(h)$ such that $\pi_i(R^k_n(h))=0$ for $0\leqq i<n$, $n<i<n+k$ and such that $\pi_n(R^k_n(h))\approx I_h$ ($n>k$, $k\leqq 6$), where I_h denotes the integers mod h.

Example 1. The case $h=0$.

$$R^1_n(0)=S^n,\quad R^2_n(0)=\{S^n|e^{n+2}_1;\quad \alpha^1=\eta_n\},$$
$$R^3_n(0)=\{R^2_n(0)|e^{n+3}_1;\quad \alpha^2=<2\bar{i}^{(1)}_{n+2}|S^n>\},$$
$$R^4_n(0)=\{R^3_n(0)|e^{n+4}_1;\quad \alpha^3=\nu_n\},$$
$$R^5_n(0)=\{R^4_n(0)|e^{n+5}_1;\quad \alpha^4=<6\bar{i}^{(1)}_{n+4},\ \bar{\eta}^{(1)}_{n+3}|R^2_n(0)>\},$$
$$R^6_n(0)=\{R^5_n(0)|e^{n+6}_1,\ e^{n+6}_2;\quad \alpha^5_1=<\bar{\nu}^{(1)}_{n+2}|S^n>,\quad \alpha^5_2=<\bar{\eta}^{(1)}_{n+4}|S^n>\},$$
$$R^7_n(0)=\{R^6_n(0)|e^{n+7}_1,\ e^{n+7}_2;\quad \alpha^6_1=<2\bar{i}^{(1)}_{n+6},\ -\bar{\nu}^{(1)}_{n+3}|R^2_n(0)>,$$
$$\alpha^6_2=<2\bar{i}^{(2)}_{n+6}|S^n\cup e^{n+4}_1>\}.$$

Example 2. The case $h=2$.

$$R^1_n(2)=\{S^n|e^{n+1}_1;2i_n\},\quad R^2_n(2)=\{R^1_n(2)|e^{n+2}_1;\alpha^1\},$$
$$R^3_n(2)=\{R^2_n(2)|e^{n+3}_1,e^{n+3}_2;\quad \alpha^2,\beta^2=<\bar{\eta}^{(1)}_{n+1}|S^n>\},$$
$$R^4_n(2)=\{R^3_n(2)|e^{n+4}_1,e^{n+4}_2;\quad \alpha^3,\beta^3=<2\bar{i}^{(2)}_{n+3},\ \bar{\eta}^{(1)}_{n+2}|R^1_n(2)>\}.$$
$$R^5_n(2)=\{R^4_n(2)|e^{n+5}_1,e^{n+5}_2;\quad <2\bar{i}^{(1)}_{n+4},\ \bar{\eta}^{(1)}_{n+3},\ 8\bar{\nu}^{(1)}_{n+1}|R^2_n(0)>,$$
$$\beta^4=<\bar{\eta}^{(1)}_{n+3},\ 3\bar{\nu}^{(1)}_{n+1}|R^2_n(0)>\},$$
$$R^6_n(2)=\{R^5_n(2)|e^{n+6}_1,e^{n+6}_2,e^{n+6}_3;\quad \alpha^5_1,\alpha^5_2,\beta^5=<2\bar{i}^{(2)}_{n+5},\bar{\eta}^{(2)}_{n+4}|R^3_n(2)>\},$$
$$R^7_n(2)=\{R^6_n(2)|e^{n+7}_1,e^{n+7}_2,e^{n+7}_3,e^{n+7}_4;\quad \alpha^6_1,\alpha^6_2,\beta^6_1=<\bar{\nu}^{(2)}_{n+3}|R^1_n(2)>,$$
$$\beta^6_2=<\bar{\eta}^{(1)}_{n+5},\ \bar{\eta}^{(2)}_{n+5}|R^4_n(0)\cup e^{n+1}_1>\}.$$

Example 3. The case $h=3$.

$$R^i_n(3)=\{S^n|e^{n+1}_1;3i_n\}\quad (i=1,2,3),$$
$$R^4_n(3)=\{R^3_n(3)|e^{n+4}_1;\alpha^3\},$$
$$R^5_n(3)=\{R^4_n(3)|e^{n+5}_1,e^{n+5}_2;\quad <3\bar{i}^{(1)}_{n+4},\ -\bar{\nu}^{(1)}_{n+1}|S^n>,\quad \gamma^4=<8\bar{\nu}^{(1)}_{n+1}|S^n>\},$$
$$R^6_n(3)=R^7_n(3)=\{R^5_n(3)|e^{n+6};\quad \gamma^5=<3\bar{i}^{(2)}_{n+5}|R^1_n(3)>\}.$$

In the above, the notations $\bar{\eta}^{(l)}_r$, $\bar{i}^{(l)}_r$ $(l=1,2)$ denote the maps $\bar{\eta}_r$: $(e^{r+1},\dot{e}^{r+1})\to(e^r_l,\dot{e}^r_l)$, $\bar{i}_r:(e^r,\dot{e}^r)\to(e^r_l,\dot{e}^r_l)$ respectively. The orders of the elements $\{\alpha^k\}\in\pi_{n+k}(R^k_n(0))$, $\{\beta^k\}\in\pi_{n+k}(R^k_n(2))$, $\{\gamma^k\}\in\pi_{n+k}(R^k_n(3))$ are as follows: The orders of $\{\alpha^k\}$ (k: even), $\{\beta^k\}$ (k: odd) and $\{\gamma^5\}$ are all zero, and the orders of $\{\alpha^3\}$, $\{\gamma^4\}$ are 6, 3 respectively. The remainders are all of order 2.

§3. Let $H^q(K,L;G)$ be the q-th cohomology group of a complex pair (K,L) with coefficients in G. Suppose that $t:G\to G'$ is a

4) J. P. Serre: Sur les groups d'Eilenberg—MacLane, C. R. (Paris) **234** (1952), and Sur la suspension de Freudenthal, ibid., **234** (1952).

H. Toda: Generalized Whitehead products and homotopy groups of spheres, to appear in the journal indicated in footnote 1.

homomorphism such that $rt=0$ $(r=2,3)$. Following N. E. Steenrod[5], we can define for any even i the cyclic reduced power

(2) $\mathrm{St}_r^i(t): H^q(K,L;G)\to H^{q+i}(K,L;G_r')$.[6]

If i is odd, we have for any homomorphism $t:G\to G'$ the cyclic reduced power $\mathrm{St}_r^i(t): H^q(K,L;G)\to H^{q+i}(K,L;G')$. Especially we can consider that $\mathrm{St}_r^1(t)$ takes values in $H^{q+i}(K,L;{}_rG')$:[6]

(3) $\mathrm{St}_r^1(t): H^q(K,L;G)\to H^{q+i}(K,L;{}_rG')$.

Modifying the definition of the cyclic reduced power, we can define[7] an operation

(4) $\overline{\mathrm{St}}_r^i(t): H^q(K,L;G)\to H^{q+i}(K,L;G')$

for any given homomorphism $t:G\to G_r'$ and any odd i. t is called a trace for the reduced power. $\mathrm{St}_r^i(t)$ satisfies the following properties: i) $f^*\mathrm{St}_r^i=\mathrm{St}_r^i f^*$ for any map f, ii) $\delta^*\mathrm{St}_r^i=\mathrm{St}_r^i\delta^*$ for the coboundary operator δ^*, iii) $(f-g)^*\mathrm{St}_r^i=\mathrm{St}_r^i(f-g)^*$ where $(f-g)^*$ denotes the difference homomorphism for maps f,g such that $f|L=g|L$, iv) $r\mathrm{St}_r^i=0$. As for $\overline{\mathrm{St}}_r^i$, we have also the similar properties.

Let Ω^8 be the quarternion projective plane and Ω^{n+4} the $(n-4)$-fold suspension of Ω^8 $(n\geqq 4)$. If we denote by $\{e^{n+k}(r)\}$ a generator of $H^{n+k}(\Omega^{n+4};I_r)$, we have

$$\mathrm{St}_r^4\{e^n(0)\}=\pm\{e^{n+4}(r)\} \qquad (r=2,3)$$

§ 4. Let $\{a\}$ be an element of $\pi_n(Y)$, and $\alpha^k: S^{n+k}\to R_n^k(0)$ the map defined in Example 1. Let us consider S^n as the n-skelton of $R_n^k(0)$, and let $a':S^n\to Y$ be a representative of a. Extend a' to $\bar a': R_n^k(0)\to Y$ and map $\{a\}$ to $\{\bar a'\circ\alpha^k\}\in\pi_{n+k}(Y)$. Then this mapping determines a homomorphism α_*^k of $\pi_n(Y)$ of $\pi_{n+k}(Y)$ if $k=1,3,5$, and to $(\pi_{n+k}(Y))_2$ if $k=2,6$, and to $(\pi_{n+k}(Y))_6$ if $k=4$. Similary, we can define a homomorphism β_*^k of ${}_2(\pi_n(Y))$ to $\pi_{n+k}(Y)$ if $k=2,4,6$, and to $(\pi_{n+k}(Y))_2$ if $k=3,5$, by using $\beta^k: S^{n+k}\to R_n^k(2)$. Furthermore we can define a homomorphism γ_*^k of ${}_3(\pi_n(Y))$ to $\pi_{n+k}(Y)$ if $k=4$, and to $(\pi_{n+k}(Y))_3$ if $k=5$, by using $\gamma^k: S^{n+k}\to R_n^k(3)$.

Finally let $i_*:\pi_n(Y)\to\pi_n(Y)$, $i_*:{}_2(\pi_n(Y))\to{}_2(\pi_n(Y))$ be the identity homomorphisms, and $p_{r*}:\pi_n(Y)\to(\pi_n(Y))_r$ the projection.

Taking $\alpha_*^k,\beta_*^k,\gamma_*^k,i_*,p_{r*}$ as a trace of (2), (3) or (4), we have the various cyclic reduced powers. Using these operations, our

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5) N. E. Steenrod: Products of cocycles and extensions of mappings, Ann. of Math., **48** (1947), and Reduced powers of cohomology classes, ibid., **56** (1952). St_r^i coincides with the Steenrod's operation $\mathfrak{P}^r{}_{(r-1)q-i}$ except the signature.

6) ${}_rG$ denotes the subgroup of G which consists of all the elements of order r, and G_r is the factor group G/rG.

7) For $r=2$, the similar operation was considered by N. Shimada—H. Uehara: Classification of mappings of an $(n+2)$-complex..., Nagoya Math. Jour., **4** (1952).

main theorems can be stated as follows:

Theorem 1 (the relative extension theorem)

Let Y be a complex with (1) *and let $n \geqq k+2$. If f, g are maps of $K^n \cup L$ into Y which coincide on L and possess extensions to $K^{n+1} \cup L$, then the difference of their secondary obstructions $z^{n+k+1}(f) - z^{n+k+1}(g)$ is given by $\pm P_k(\lambda^n)$, where $\lambda^n = (f-g)^* \mathbf{1}$*[8] *and P_k is the following:*

$$P_k = \mathrm{St}_2^2(\alpha_*^1) \text{ if } k=1,^{9)} \quad = \overline{\mathrm{St}}_2^3(\alpha_*^2) + \mathrm{St}_2^2(\beta_*^2)\mathrm{St}_2^1(i_*) \text{ if } k=2,^{10)}$$

$$= \mathrm{St}_2^4(3\alpha_*^3) + \overline{\mathrm{St}}_2^3(\beta_*^3)\mathrm{St}_2^1(i_*) + \mathrm{St}_3^4(-2\alpha_*^3) \quad \text{if } k=3,^{11)}$$

$$(5) \quad = \overline{\mathrm{St}}_2^5(\alpha_*^4) + \mathrm{St}_2^4(\beta_*^4)\mathrm{St}_2^1(i_*) + \overline{\mathrm{St}}_3^5(-\alpha_*^4) + \mathrm{St}_3^4(\gamma_*^4)\mathrm{St}_3^1(i_*) \quad \text{if } k=4,$$

$$= \mathrm{St}_2^4(\alpha_{1*}^5)\mathrm{St}_2^2(p_{2*}) + \mathrm{St}_2^2(\alpha_{2*}^5)\mathrm{St}_2^4(p_{2*}) + \overline{\mathrm{St}}_2^5(\beta_*^5)\mathrm{St}_2^1(i_*) + \overline{\mathrm{St}}_3^5(\gamma_*^5)\mathrm{St}_3^1(i_*) \quad \text{if } k=5,$$

$$= \overline{\mathrm{St}}_2^5(\alpha_{1*}^6)\mathrm{St}_2^2(p_{2*}) + \overline{\mathrm{St}}_2^3(\alpha_{2*}^6)\mathrm{St}_2^4(p_{2*}) + \mathrm{St}_2^4(\beta_{1*}^6)\mathrm{St}_2^2(i_*)\mathrm{St}_2^1(i_*) + \mathrm{St}_2^2(\beta_{2*}^6)\mathrm{St}_2^4(i_*)\mathrm{St}_2^1(i_*) \quad \text{if } k=6.$$

Theorem 2 (the homotopy classification theorem)

Let Y be a complex with (1) *and $f_0, f_1 : K^{n+1} \to Y$ be normal maps such that $f_0|K^n = f_1|K^n$ $(n \geqq k+2)$. Then $f_0 \simeq f_1$ rel. L if and only if there exists a cohomology class $e^{n-1} \in H^{n-1}(K, L; \pi_n(Y))$ such that $d^{n+k}(f_0, f_1) = \pm P_k e^{n-1}$, where $d^{n+k}(f_0, f_1) \in H^{n+k}(K, L; \pi_{n+k}(Y))$ is the cohomology class of the separation cocycle for f_0 and f_1 and P_k is the homomorphism* (5).

The proofs of Theorems 1 and 2 are performed by a method which is in a sense a generalization of that of Steenrod[5]: the complexes M^n in his paper are replaced here by the cell complexes $R_n^k(h)$.

§5. J. Adem[12] recently obtained the various relations with respect to the iterated Steenrod squares St_2^i. We can prove some of these relations by using the cell complex $R_n^k(2)$ in Example 2. Among all, we have

$$\mathrm{St}_2^2\mathrm{St}_2^2 = \mathrm{St}_2^3\mathrm{St}_2^1, \quad \mathrm{St}_2^4\mathrm{St}_2^1 + \mathrm{St}_2^5 = \mathrm{St}_2^2\mathrm{St}_2^3,$$

with coefficients in I_2.

8) **1** denotes the same as in J. H. C. Whitehead: On the theory of obstructions, Ann. of Math., **54** (1951).

9) This was obtained by many authors: J. H. C. Whitehead[8], S. Eilenberg—S. MacLane[2], and M. M. Postnikov (in Russian).

10) This was obtained first by N. Shimada—H. Uehara[7].

11) We can get also in this case the theorems 1 and 2 for $n=k+1=4$. The Pontrjagin square and Postnikov square are needed. cf.[1]

12) J. Adem: The iteration of the Steenrod Squares in algebraic topology, Proc. Nat. Acad. of Sci., U.S.A., **38** (1952).