

100. On the Isomorphism of Topological Groups

By Takashi KASUGA

Department of Mathematics, Osaka University

(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1953)

In this note we shall prove some simple theorems on the identification of two topological groups with the same underlying abstract group.

1. Some notations and definitions. In the following we mean by the word "topology" a topology which satisfies Hausdorff's axioms.

We denote by $R(T)$ a set R with topology T (in this note R may be an abstract set or an abstract group or an abstract linear space). For two sets R_1 and R_2 (without topologies) we denote by $R_1 \times R_2$ their direct product, that is, the set of all pairs (x_1, x_2) where $x_1 \in R_1$ and $x_2 \in R_2$. When R_1 and R_2 are both abstract groups or both abstract linear spaces, we can consider $R_1 \times R_2$ as an abstract group or as an abstract linear space in the well-known manner (in the case of groups, we define $(x_1, x_2)(y_1, y_2)$ by (x_1y_1, x_2y_2) where $x_1, y_1 \in R_1$ and $x_2, y_2 \in R_2$, and in the case of linear spaces, we define $(x_1, x_2) + (y_1, y_2)$ and $\alpha(x_1, x_2)$ by $(x_1 + y_1, x_2 + y_2)$ and $(\alpha x_1, \alpha x_2)$ respectively where $x_1, y_1 \in R_1$, $x_2, y_2 \in R_2$ and α is any real number). When $R_1(T_1)$ and $R_2(T_2)$ are two topological spaces, we denote by $R_1(T_1) \times R_2(T_2)$ the so-called topological direct product of $R_1(T_1)$ and $R_2(T_2)$. We denote the topology of the topological space $R_1(T_1) \times R_2(T_2)$ by $T_1 \times T_2$. Evidently by the definitions $R_1 \times R_2(T_1 \times T_2) = R_1(T_1) \times R_2(T_2)$. For a subset S of a topological space $R(T)$, we denote by $S\{T\}$ S with the topology induced by T . If R is endowed with two topologies T and T^* , and T is stronger (that is, with more open sets) than T^* or at least equivalent to T^* , then we write $T \geq T^*$. By Δ_R , we denote the diagonal of $R \times R$, that is, the set of the elements of $R \times R$ which are of the form (a, a) where $a \in R$. When R is an abstract group or an abstract linear space, Δ_R is a subgroup or a linear subspace of $R \times R$ respectively.

In the following, we shall say that a topological space is semi-compact, if it is locally bicomact and can be represented as a sum of a number, countable at most, of bicomact sets.

2. We prove first a simple lemma.

Lemma 1. *If R is endowed with three topologies T_1, T_2, T^* and $T_1 \geq T^*, T_2 \geq T^*$, then Δ_R is closed in $R(T_1) \times R(T_2)$.*

Proof. $T_1 \times T_2 \geq T^* \times T^*$, since $T_1 \geq T^*$ and $T_2 \geq T^*$. On the other hand, Δ_R is closed in $R \times R(T^* \times T^*) (= R(T^*) \times R(T^*))$, as

$R(T^*)$ is a Hausdorff space. Hence Δ_R is closed in $R \times R(T_1 \times T_2)$ ($= R(T_1) \times R(T_2)$).

3. Theorem 1. *Let $G(T_1)$ and $G(T_2)$ be both semi-compact topological groups with the same underlying abstract group G . If we can define a topology T^* on G such that $T_1 \geq T^*$ and $T_2 \geq T^*$ ($G(T^*)$ needs not be necessarily a topological group), then $G(T_1)$ and $G(T_2)$ are one and the same topological group.*

Proof. From the assumption that $G(T_1)$ and $G(T_2)$ are both semi-compact topological groups, we can easily prove that $G(T_1) \times G(T_2)$ is also a semi-compact topological group. On the other hand, by Lemma 1, Δ_G is closed in $G(T_1) \times G(T_2)$ as $T_1 \geq T^*$ and $T_2 \geq T^*$. Hence $\Delta_G\{T_1 \times T_2\}$ (that is, Δ_G with the topology induced by the topology $T_1 \times T_2$ of $G(T_1) \times G(T_2)$) is a semi-compact topological group.

The mapping $(a, a) \rightarrow a$, where $a \in G$, is continuous as a mapping of $\Delta_G\{T_1 \times T_2\}$ onto $G(T_1)$ as well as, as a mapping of $\Delta_G\{T_1 \times T_2\}$ onto $G(T_2)$ and at the same time it is an abstract group isomorphism of Δ_G onto G . It is well-known that if an abstract group homomorphism of a semi-compact topological group onto another semi-compact topological group is continuous, then this mapping is open¹⁾. Since $G(T_1)$, $G(T_2)$ and $\Delta_G\{T_1 \times T_2\}$ are semi-compact topological groups, the mapping $(a, a) \rightarrow a$ where $a \in G$, is an isomorphism of $\Delta_G\{T_1 \times T_2\}$ onto $G(T_1)$ and is at the same time an isomorphism of $\Delta_G\{T_1 \times T_2\}$ onto $G(T_2)$. Hence the identity mapping $a \rightarrow a$ where $a \in G$, is an isomorphism of $G(T_1)$ onto $G(T_2)$. Thus Theorem 1 is proved.

4. From Theorem 1 we can easily deduce the two following theorems.

Theorem 2. *Let $R(T')$ be a topological space and G be a group of transformations of R as an abstract set (we denote by $g(p)$ the result of the transformation $g(\in G)$ as applied to $p(\in R)$). If we can define a topology T on G such that $G(T)$ is a semi-compact topological group and the mapping $A_p g \rightarrow g(p)$ where $g \in G$ and p is a fixed element of R , is continuous as a mapping of $G(T)$ into $R(T')$ for any fixed element p of R , then such T is uniquely determined.*

Proof. We can easily construct the weakest topology T^* among the topologies T^{**} of G such that the mapping $A_p g \rightarrow g(p)$ is continuous as a mapping of $G(T^{**})$ into $R(T')$ for any fixed element p of R ($G(T^{**})$ needs not be necessarily a topological group).

When p_1, \dots, p_n are elements of R and g_0 is an element of G and U_1, \dots, U_n are neighbourhoods in $R(T')$ of $g_0(p_1), \dots, g_0(p_n)$ respectively, we denote by $V(g_0: p_1, \dots, p_n: U_1, \dots, U_n)$ the set of the elements g of G such that $g(p_i) \in U_i$, $i = 1, \dots, n$.

We can define the topology T^* of G by taking $V(g_0: p_1, \dots, p_n: U_1, \dots, U_n)$ for any finite number of elements p_1, \dots, p_n of R and for any neighbourhoods U_1, \dots, U_n in $R(T')$ of $g_0(p_1), \dots, g_0(p_n)$ respectively as the neighbourhoods of any element g_0 of G . Hausdorff's axioms can be easily verified for this system of neighbourhoods. Then for any T which satisfies the conditions of Theorem 2, $T \geq T^*$. Hence by Theorem 1, T is uniquely determined.

Theorem 3. *Let H be a subgroup of an abstract group G and $G(T')$ be a topological group with the underlying abstract group G . If we can define on H a topology T such that $H(T)$ is a semi-compact topological group and the identity mapping: $h \rightarrow h$ where $h \in H$, is continuous as a mapping of $H(T)$ into $G(T')$, then T is uniquely determined.*

Proof. If we denote by T^* the topology induced on H by the topology T' of $G(T')$, then for any T which satisfies the conditions of Theorem 3, $T \geq T^*$. Hence by Theorem 1, T is uniquely determined.

5. Theorem 4. *Let $L(N_1)$ and $L(N_2)$ be two Banach spaces with the norms N_1 and N_2 respectively, but with the same underlying abstract linear space L . If the weak topologies of $L(N_1)$ and $L(N_2)$ are equivalent, then $L(N_1)$ and $L(N_2)$ are isomorphic by the identity mapping. (Roughly speaking, a Banach space is determined in its topological structure by its algebraic structure and its weak topology.)*

Proof. We denote by T_1 and T_2 the topologies on L induced by the norms N_1 and N_2 respectively and by W the common weak topology of $L(N_1)$ and $L(N_2)$. By defining a suitable norm (not uniquely determined) on $L \times L$, we can consider the linear space $L \times L$ as a Banach space whose topology is $T_1 \times T_2$. We denote this norm by $N_1 \times N_2$ and this Banach space by $L \times L(N_1 \times N_2)$ or by $L(N_1) \times L(N_2)$.

By Lemma 1, \mathcal{A}_L is a closed linear subspace of $L(N_1) \times L(N_2)$ as $T_1 \geq W$ and $T_2 \geq W$. Hence we can consider \mathcal{A}_L as a Banach space with the norm induced on \mathcal{A}_L by the norm $N_1 \times N_2$ of $L(N_1) \times L(N_2)$. We denote this Banach space by $\mathcal{A}_L \{N_1 \times N_2\}$.

The mapping $(a, a) \rightarrow a$ where $a \in L$, is continuous as a mapping of $\mathcal{A}_L \{N_1 \times N_2\}$ onto $L(N_1)$, as well as, as a mapping of $\mathcal{A}_L \{N_1 \times N_2\}$ onto $L(N_2)$ and at the same time, it is an abstract linear isomorphism of \mathcal{A}_L onto L . Moreover $\mathcal{A}_L \{N_1 \times N_2\}$, $L(N_1)$ and $L(N_2)$ are all Banach spaces. Hence by a well-known theorem of Banach²⁾, the mapping $(a, a) \rightarrow a$ where $a \in L$, is an isomorphism of $\mathcal{A}_L \{N_1 \times N_2\}$ onto $L(N_1)$ and is at the same time an isomorphism of $\mathcal{A}_L \{N_1 \times N_2\}$ onto $L(N_2)$. Then the mapping $a \rightarrow a$ where $a \in L$, is an isomorphism of $L(N_1)$ onto $L(N_2)$. Thus Theorem 4 is proved.

Notes

1) Cf. T. Tannaka, *Isogunron* (in Jap.), 36-37 (1949). Also cf. H. Freudenthal, *Einige Sätze über topologische Gruppen*, *Ann. Math.*, **37** (1936), where this proposition is proved, under the assumption that both topological groups are separable.

2) Cf. Banach, *Théorie des opérations linéaires*, 41 (1932).