98. *Positive Linear Functionals on Self-Adjoint B-Algebras*

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1. A self-adjoint Banach (B-) algebra (or abbrev. *B_*-algebra*) A is a B-algebra over the complex scalar field K which admits such a^*-operation as is a conjugate linear, involutory, anti-automorphism of A, i.e. $(\alpha a+b)^*=\bar{\alpha}a^*+b^*$, $a^{**}=a^*$, and $(ab)^*=b^*a^*$ for $a, b\in A$, $\alpha\in K$.

If a B-algebra A has an approximate identity $\{e^\lambda\}$, $ae^\lambda\longrightarrow a$ and $e^\lambda a\longrightarrow a$ (strongly), we call A *semi-unitary*, and if A has identity e (of norm 1), *unitary*.

The collection of all hermitian elements, $a^*=a$, of A is denoted by $H(A)$ and called the *hermitian kernel* of A; $H(A)$ forms a normed sub-space of A, and if A is commutative, a sub-B-algebra which is necessarily real.

A B_*-algebra possessing an additional condition $\|a^*a\|=\|a\|\cdot\|a^*\|$ is *a* B^*-algebra in the sense of R.V. Kadison[1].

A commutative real B-algebra is always regarded as *a* B_* algebra, over the reals, with $A=H(A)$.

2. For a real commutative unitary B-algebra, i.e. unitary A with $A=H(A)$, the following assertion is a well-known fact:

The set Π of real linear functionals on A which is non-negative on squares and 1 *on e is a w^*-compact,*[2] *convex set.*

If Π is non-void, each of its extreme points is a multiplicative linear functional, so that $f^{-1}(0)$ is a maximal ideal of A for every extreme f of Π.[3]

In this note, we intend to pursue the relations between extreme points of Π and maximal ideals in the case of non-commutative B_*-algebras A_*.

We begin with some notations:

$\Gamma(\cdot)=$ the dual space of a normed vector space $(\cdot)$.

$\Xi(A_*)=$the sub-space of $\Gamma(A_*)$, over the reals, whose elements satisfy $f(a^*)=\overline{f(a)}$.

$\hat{\Pi}(A_*)=$the convex subset of $\Xi(A_*)$, such that $f(a^*a)\geqq 0$.

$\Phi(A_*)=$the set of all multiplicative linear functionals on A_*;

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1) A representation theory for commutative topological algebra, Memoirs of Amer. Math. Soc., **7** (1951).

2) With respect to the weak topology as functionals.

3) R.V. Kadison, loc. cit., pp. 23–24.

$\hat{\Phi}(A_*)=\Phi(A_*)\cap\Xi(A_*)$, which is clearly $\subset\hat{\Pi}(A_*)$.

Proposition 1. For any $f\in\Gamma(A_*)$, *the set* $'\mathfrak{J}_f$, $\mathfrak{J}'_f$, *or* $\mathfrak{J}_f$,

$$'\mathfrak{J}_f=\{a\,;\,f(xa)=0 \text{ for every } x \text{ of } A_*\},$$
$$\mathfrak{J}'_f=\{a\,;\,f(ax)=0 \text{ for every } x \text{ of } A_*\},$$
$$\mathfrak{J}_f=\{a\,;\,f(xay)=0 \text{ for every } x,y \text{ of } A_*\},$$

forms a closed left, right, or two-sided ideal of A_* *respectively; if* A_* *is semi-unitary,* $f(a)=0$ *for* $a\in{}'\mathfrak{J}_f$, $\mathfrak{J}'_f$, *or* $\mathfrak{J}_f$.

Proposition 2. For any $f\in\hat{\Pi}(A_*)$, *the quotient B-space* $A_*/'\mathfrak{J}_f$ *(or* $A_*/\mathfrak{J}'_f$*) forms a pre-Hilbert space with the inner product*

$$(2.1)\qquad (X_a, X_b)_f=f(b^*a)\qquad (or =f(ab^*)),$$

where X_a *is a residue class containig* a*; the completion of* $A_*/\mathfrak{J}^f$ *with respect to the norm* $\|X_a\|=(X_a, X_a)^{1/2}$ *is a Hilbert algebra.*

The Hilbert space, completed from $A_*/'\mathfrak{J}_f$ (or $A_*/\mathfrak{J}'_f$), is denoted by $'\mathfrak{H}_f$ (or resp. $\mathfrak{H}'_f$).

Proposition 3. We have $\Phi(A_*)=\hat{\Phi}(A_*)$, *and for any* $\varphi\in\Phi(A_*)$, *the set* $\mathfrak{J}_\varphi=\varphi^{-1}(0)$ *forms a maximal two-sided regular ideal such that* $A_*/\mathfrak{J}_\varphi\cong K$.

For the proof of Prop. 2, generalized Cauchy-Schwarz's lemma, $|f(ab^*)|^2=|f(ba^*)|^2\leqq f(aa^*)\,f(bb^*)$, is usefull.

3. Next, we shall define two manners of product in $H(A_*)$; 1) Jordan product

$$(3.1)\qquad a\circ b=\frac{1}{2}(ab+ba),$$

which is always commutative, distributive, but non-associative, and $a\circ a=a^2$.

2) Special Poisson's product

$$(3.2)\qquad [a,b]=\frac{1}{2i}(ab-ba),$$

which is skew-symmetric, distributive, and satisfies the Jacobi's equality. The set of all $[a,b]$, $a,b\in H(A_*)$ is denoted by $W(A_*)$, which is contained in $H(A_*)$.

If A_* is commutative, $a\circ b=ab=ba$ and $W(A_*)\equiv(0)$.

$\tilde{\Pi}(H(A_*))$=the convex subset of $\Gamma(H(A_*))$ consisting of all such functionals;

$$(3.3)\qquad 2|f([a,b])|\leqq f(a^2)+f(b^2),\ a,b\in H(A_*).$$

$\tilde{\Phi}(H(A_*))$=all of multiplicative linear functionals on $H(A_*)$ with respect to the product (3.1), vanishing on $W(A_*)$.

Theorem 1. $\Xi(A_*)\cong\Gamma(H(A_*))$, $\hat{\Pi}(A_*)\cong\tilde{\Pi}(H(A_*))$ *and* $\Phi(A_*)\cong\tilde{\Phi}(H(A_*))$, *where the sign "$\cong$" means a topological isomorphism in which the restriction on* $H(A_*)$ *of each element in the left coincides with the corresponding one in the right.*

In virtue of this Theorem, the Krein-Milman's "extreme points" theorem is also valid for a bounded, regularly convex set in $\Xi(A_*)$ even in the case of complex algebra; denoting the unit sphere of $\Gamma(A_*)$ (or $\Gamma(H(A_*))$) by E (or resp. E_0), $E \frown \Xi$ is w^*-compact, which is a modified formula of Kakutani-Dieudonné's theorem.

4. Now, $\widetilde{\Pi}(H(A_*))$ is w^*-closed in $\Gamma(H(A_*))$, so that $\widetilde{E}_0 = E_0 \frown \widetilde{\Pi}(H(A_*))$ is w^*-compact and regularly convex, then it has the set $S(\widetilde{E}_0)$ of extreme points whose convex hull is dense in $\widetilde{E}_0$; if $'\mathfrak{J}_f = '\mathfrak{J}_g$, we write $\hat{f} \sim \hat{g}$, calling them *equivalent*[4] $\hat{f} \sim \hat{g}$ yields $\hat{f} \sim (\alpha \hat{f} + \beta \hat{g})$, for $\alpha, \beta \geqq 0$ with $\alpha + \beta = 1$.

Definition. If $\hat{f} \in \widetilde{E}_0$ is equivalent to no linear convex combination of $\hat{g}$ and $\hat{h}$, each of which is in E_0 and not equivalent to $\hat{f}$, then $\hat{f}$ is said to be *weakly extreme* (*w. extr.*) *in* $\widetilde{E}_0$.

From the definition, it follows immediately:

i) If $\hat{f}$ is *w.* extr. in $\widetilde{E}_0$ and if $\hat{f} \sim \hat{g}$, then $\hat{g}$ is also *w.* extr. in $\widetilde{E}_0$,

ii) $'\mathfrak{J}_f \frown '\mathfrak{J}_g \subset '\mathfrak{J}_{\alpha f + \beta g}$ for $\alpha, \beta \geqq 0$, $\alpha + \beta = 1$.

It is not sure whether an extreme point of $\widetilde{E}_0$ is *w.* extr. or not in general cases. But we can settle an important result:

Theorem 2. For a semi-unitary B_*-algebra, a necessary and sufficient condition that $'\mathfrak{J}_f$ would be maximal is that $\hat{f}$ is w. extr. in $\widetilde{E}_0$ for $\|f\| \leqq 1$.

To prove this, we make use of the Hilbert space completed from $A_*/\mathfrak{J}_f$ and orthogonal decomposition in it.

The above assertions are also valid for $\mathfrak{J}'_f$ or $\mathfrak{J}_f$ at all. We shall define another notion;

Definition. If i) $'\mathfrak{J}_f$, $\mathfrak{J}'_f$, or $\mathfrak{J}_f$ is a regular ideal and ii) $f(j) = 1$ for an identity j modulo the corresponding ideal, then f (or $\hat{f}$) is called *left, right, or two-sided regular*; but we need essentially only two regularities of f, *one-sided and two-sided,* since if f is left regular having an identity j modulo $'\mathfrak{J}_f$, then f is also right regular, having an identity j^* modulo $\mathfrak{J}'_f = ('\mathfrak{J}_f)^*$.

The set intersection of $\widetilde{E}_0$ and of all one-sided (two-sided) regular functionals is denoted by $\hat{E}_0$ (resp. $\hat{\hat{E}}_0$), which is evidently convex and w^*-closed.

If A_* is unitary, it holds $\hat{E}_0 = \hat{\hat{E}}_0$, each of whose element is called a "state" in the case of C^*-algebra.[5]

Theorem 3. For $\hat{f}$ in $\hat{E}_0$ (or $\hat{\hat{E}}_0$), if $'\mathfrak{J}_f$ (resp. $\mathfrak{J}_f$) is not maximal, then there exists a segment in $\hat{E}_0$ (resp. $\hat{\hat{E}}_0$) just in which $\hat{f}$ is an inner point.

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4) $\hat{f}$ is a corresponding element of $\widetilde{\Pi}(H(A))$ to f of $\hat{\Pi}(A_*)$ with respect to the isomorphism in Thr. 1; $f = \hat{f}$ on $H(A_*)$.

5) See, I. E. Segal, Two-sided ideals in operator algebras, Ann. Math., **50** (1949).

Corollary 3. 1. For an extreme $\hat{f}$ in $\hat{E}_0$ (or $\hat{\hat{E}}_0$), each of $'\mathfrak{J}_f$ and $\mathfrak{J}'_f$ (or resp. $\mathfrak{J}_f$) is maximal and regular.

By means of this Corollary and Thr. 2, we have

Corollary 3. 2. Every extreme point of $\hat{E}_0$ (or, if the algebra is unitary, of $\tilde{E}_0$) is w. extr. in it.

Theorem 4. Every non-null φ in Φ (A_*) (i.e. multiplicative) is an extreme point of $\hat{E}_0$ and of $\hat{\hat{E}}_0$.

In Thr. 3~Thr. 4, A_* is assumed to be semi-unitary.

Theorem 5. If A_* is commutative and unitary, it holds

$$extr.\ \hat{E}_0 = extr.\ \hat{\hat{E}}_0 = S(\tilde{E}_0) = \Phi^0(A_*)\,,$$

where Φ^0 means the collection of non-zero elements of Φ.

5. Assume that A_* is the group-algebra on a LC group G, then $\tilde{E}_0=\hat{E}_0$ and $\tilde{E}_0$ is one-to-one corresponding to the collection of all continuous positive definite (c.p.d.) function on G with norms less than 1, by the relation

$$f(a) = \int_G \overline{\xi(x)}\, a(x)dx, \qquad \text{for} \quad a \in A_*\,, \quad f \in E_0\,,$$

and $\xi(\cdot)$ is c.p.d. on G with $\|\xi\| = \sup_{x\in G} |\xi(x)| \leqq 1$.

In the case, every extreme $\hat{f}_0$ corresponds to an elementary c.p.d. function and all w. extr. points f consists of a segment combining each $\hat{f}_0$ and 0, i.e. $\hat{f}=\lambda\hat{f}_0$ for $0<\lambda\leqq 1$.[6]

6) See, for example, R. Godement, Les fonctions de type positif et la théorie des groupes, Trans. Amer. Math. Soc., **63** (1948).