109. Note on Dirichlet Series. XI.
On the Analogy between Singularities and Order-curves

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(Comm. by Z. SUETUNA, M.J.A., Nov. 12, 1953)

(1) **Introduction.** Let us put

(1.1) $F(s)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty).$

O. Szász has proved the next theorem, which is a generalization of Hurwitz-Pólya's theorem (E. Landau[1], p. 36).

O. Szász's Theorem (O. Szász[2], p. 107). *Let* (1.1) *have the finite simple convergence-abscissa* σ_s. *If* $\lim_{n\to+\infty} \log n/\lambda_n=0$, *then there exists a sequence* $\{\varepsilon_n\}$ $(\varepsilon_n=\pm 1)$ *such that* $\sum_{n=1}^{\infty} a_n \varepsilon_n \exp(-\lambda_n s)$ *has* $\sigma=\sigma_s$ *as the natural boundary.*

The author proved recently the following theorem of the same type:

Theorem (C. Tanaka[3], p. 308). *Let* (1.1) *have the finite simple convergence-abscissa* σ_s. *If* $\lim_{n\to+\infty} \log n/\lambda_n=0$, *then there exists a Dirichlet series* $\sum_{n=1}^{\infty} b_n \exp(-\lambda_n s)$ *having* $\sigma=\sigma_s$ *as the natural boundary such that*

(a) $|b_n|=|a_n|$ $(n=1,2,\ldots)$ *and* $\lim_{n\to+\infty}|\arg(a_n)-\arg(b_n)|=0$

or

(b) $\arg(b_n)=\arg(a_n)$ $(n=1,2,\ldots)$ *and* $\lim_{n\to+\infty}|b_n/a_n|=1$.

In this note, we shall establish analogous theorems concerning order-curves. We first begin with

Definition. *Let* (1.1) *be uniformly convergent in the whole plane. Then, we call the analytic curve* C *extending to* $\sigma=-\infty$ *the order-curve of* (1.1), *provided that, in* $D(\varepsilon; C)$ (ε: *any positive constant*), (1.1) *has the same order as in the whole plane, where* $D(\varepsilon; C)$ *is the curved strip generated by circles with radii* ε *and having its centres on* C.

Our theorems read as follows:

Theorem I. *Let* (1.1) *with* $\overline{\lim}_{n\to+\infty} \log n/\lambda_n<+\infty$ *be simply (necessarily absolutely) convergent in the whole plane, and* C *be any given analytic curve extending to* $\sigma=-\infty$. *Then, there exists a everywhere absolutely convergent Dirichlet series* $\sum_{n=1}^{\infty}\varepsilon_n a_n \exp(-\lambda_n s)$ $(\varepsilon_n=\pm 1)$, *such that it has every curve* C_τ $(-\infty<\tau<+\infty)$ *as its order-curve, where* C_τ *is obtained from moving* C *in parallel by* $i\tau$ $(-\infty<\tau<+\infty)$.

Theorem II. *Under the same assumptions as above, there exists* *a everywhere absolutely convergent Dirichlet series* $\sum_{n=1}^{\infty} b_n \exp(-\lambda_n s)$ *having every curve* C_τ $(-\infty<\tau<+\infty)$ *as its order-curve such that*

$$\text{(a)} \quad \underset{(n=1,2,\dots)}{|b_n|=|a_n|}, \ \lim_{n\to+\infty} |\arg(b_n)-\arg(a_n)|=0,$$

or

$$\text{(b)} \quad \underset{(n=1,2,\dots)}{\arg(b_n)=\arg(a_n)}, \ \lim_{n\to+\infty} |b_n/a_n|=1.$$

(2) **Proof of Theorem I.** Let σ_s, σ_a be the simple and absolute convergence-abscissa of (1. 1) respectively. By the well-known theorem (D. V. Widder[4], p. 49), we have

$$0 \leqq \sigma_a - \sigma_s \leqq \overline{\lim_{n\to+\infty}} \log n/\lambda_n,$$

so that, from $\sigma_s=-\infty$ and $\overline{\lim}_{n\to+\infty} \log n/\lambda_n<+\infty$, $\sigma_a=-\infty$ immediately follows. Hence (1. 1) is necessarily absolutely convergent in the whole plane.

Let (1. 1) be of order ρ. Then, by J. Ritt's theorem[5] and $\overline{\lim}_{n\to+\infty}$ $\log n/\lambda_n <+\infty$, we have

$$(2.1) \qquad \overline{\lim_{n\to+\infty}} 1/\lambda_n \log \lambda_n . \ \log|a_n|=-1/\rho.$$

Hence we can select from $\{\lambda_n\}$ a sequence $\{\lambda_{ni}\}$ such that

$$(2.2) \qquad \begin{cases} \text{(i)} \ \lim_{i\to+\infty} 1/\lambda_{ni} \log \lambda_{ni} . \ \log|a_{ni}|=-1/\rho, \\ \text{(ii)} \ \underline{\lim}_{i\to+\infty} (\lambda_{ni+1}-\lambda_{ni})>0, \ \lim_{i\to+\infty} i/\lambda_{ni}=0. \end{cases}$$

Now let us put

$$\begin{aligned} F(s) &= \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \\ &= \sum_{n\neq(ni)} a_n \exp(-\lambda_n s) + \sum_{i=1}^{\infty} a_{ni} \exp(-\lambda_{ni} s) \\ &= f_0(s) + R(s). \end{aligned}$$

Since $F(s)$ converges absolutely everywhere, $R(s)$ is evidently absolutely convergent. Taking account of J. Ritt's theorem and (2. 2), $R(s)$ is also of order ρ.

Next put

$$R(s)=\sum_{i=1}^{\infty} a_{ni} \exp(-\lambda_{ni}s) = f_1(s)+f_2(s)+\cdots+f_n(s)+\cdots,$$

where $f_n(s)$ $(n=1,2,\dots)$ is a Dirichlet series having infinite number of terms of $R(s)$, which is also everywhere absolutely convergent. We define new Dirichlet series

$$F(s;\{\varepsilon_n\}) = f_0(s)+\varepsilon_1 f_1(s)+\varepsilon_2 f_2(s)\cdots+\varepsilon_n f_n(s)+\cdots,$$

where $\varepsilon_n=\pm 1$ $(n=1,2,\dots)$. Since $F(s)$ converges absolutely everywhere, $F(s;\{\varepsilon_n\})$ is evidently everywhere absolutely convergent and by (2. 1), it is also of order ρ.

Putting

$$G(s)=F(s;\{\varepsilon_n\})-F(s;\{\varepsilon_n'\}) \quad (\{\varepsilon_n\}\neq\{\varepsilon_n'\}),$$

we can prove that

$$(2.3) \qquad \begin{cases} \text{(i)} \ G(s) \text{ is an integral function of order } \rho, \\ \text{(ii)} \ G(s) \text{ has every curve } C_\tau \ (-\infty<\tau<+\infty) \text{ as its order-curve.} \end{cases}$$

In fact, setting

$$G(s)=\sum_{\nu=1}^{\infty}(\varepsilon_\nu-\varepsilon'_\nu)f_\nu(s)=\sum_{i=1}^{\infty}b_{mi}\exp(-\lambda_{mi}s),$$

we have

$$\begin{cases}\text{(i)} & m_i\in\{n_i\},\\ \text{(ii)} & |b_{mi}|=2\,|a_{mi}|,\end{cases}$$

so that, by (2. 2)

$$\text{(2. 4)}\qquad\begin{cases}\text{(i)} & \lim_{i\to+\infty}1/\lambda_{mi}\log\lambda_{mi}\cdot\ \log|b_{mi}|=-1/\rho,\\ \text{(ii)} & \varliminf_{i\to+\infty}(\lambda_{mi+1}-\lambda_{mi})>0,\quad \lim_{i\to+\infty}i/\lambda_{mi}=0.\end{cases}$$

Hence, by (2. 4), J. Ritt's theorem and an extension of G. Pólya's theorem[6], (2. 3) holds.

Let $F(s;\{\varepsilon_n\})$ have the subset $\{C_{\tau'}\}$ of $\{C_\tau\}$ as its order-curves. Corresponding to $\{C_{\tau'}\}$, we consider the set $\{\tau'\}$, which is evidently closed. Let us denote by $E(\{\varepsilon_n\})$ the complementary set of $\{\tau'\}$, which is obviously an open set. Then we can easily prove that

$$\text{(2. 5)}\qquad E(\{\varepsilon_n\})\frown E(\{\varepsilon'_n\})=0\quad\text{for}\quad\{\varepsilon_n\}\neq\{\varepsilon'_n\}.$$

In fact, if there should exist one curve $C_{\tau''}$ such that

$$\tau''\in E(\{\varepsilon_n\})\frown E(\{\varepsilon'_n\})\neq 0,$$

then $C_{\tau''}$ would not be the order-curve of $G(s)$. For, in $D(\varepsilon;C_{\tau''})$ (ε: arbitrary positive constant), $F(s;\{\varepsilon_n\})$ and $F(s;\{\varepsilon'_n\})$ have the order less than ρ, so that $G(s)=F(s;\{\varepsilon_n\})-F(s;\{\varepsilon'_n\})$ has also the order less than ρ in this curved-strip, i.e. $C_{\tau''}$ is not the order-curve of $G(s)$, taking account of (2. 3) (i). On the other hand, by (2. 3) (ii), $G(s)$ has all curves as its order-curves, which contradicts the existence of $C_{\tau''}$. Thus, (2. 5) is proved.

If $E(\{\varepsilon_n\})\neq 0$ for all $\{\varepsilon_n\}$, by (2. 5) the function-family $\{F(s;\{\varepsilon_n\})\}$ is at most of enumerable power, which contradicts the power of continuum of $\{F(s;\{\varepsilon_n\})\}$. Hence, for at least one $\{\varepsilon_n\}$, $E(\{\varepsilon_n\})=0$. In other words, $F(s;\{\varepsilon_n\})$ has every curve C_τ $(-\infty<\tau<+\infty)$ as its order-curve and it is evidently of the form

$$\sum_{n=1}^{\infty}\varepsilon'_n a_n\exp(-\lambda_n s),\quad \varepsilon'_n=\pm1,$$

which is to be proved.

(3) **Proof of Theorem II.** By the arguments as above, $F(s)$ is everywhere absolutely convergent. Let $F(s)$ be of order ρ. Let us put

$$F(s;\theta,\alpha)$$
$$=\sum_{i=1}^{\infty}a_{ni}\exp(\alpha\theta/\lambda_{ni}).\ \exp(-\lambda_{ni}s)+\sum_{n\neq\{ni\}}a_n\exp(-\lambda_n s),$$

where the sequence $\{\lambda_{ni}\}$ is determined by (2. 2), and α,θ (θ: real) are constants determined later. Then $F(s;\theta,\alpha)$ is an integral function of order ρ, taking account of J. Ritt's theorem and (2. 2).

Putting

$$G(s)=F(s;\theta_1,\alpha)-F(s;\theta_2,\alpha)\quad(\theta_1\neq\theta_2),$$

$G(s)$ is also an integral function of order ρ. For,

$$G(s)=\sum_{i=1}^{\infty}a_{ni}\{\exp(\alpha\theta_1/\lambda_{ni})-\exp(\alpha\theta_2/\lambda_{ni})\}\cdot\exp(-\lambda_{ni}s)$$
$$=\sum_{i=1}^{\infty}a_{ni}\ 0(1/\lambda_{ni}).\ \exp(-\lambda_{ni}s),$$

so that, by (2. 2)

$$\overline{\lim_{i\to+\infty}}\ 1/\lambda_{ni}\log\lambda_{ni}.\quad\log|a_{ni}\,0(1/\lambda_{ni})|$$
$$=\lim_{i\to+\infty}1/\lambda_{ni}\log\lambda_{ni}.\ \log|a_{ni}|+\lim_{i\to+\infty}1/\lambda_{ni}\log\lambda_{ni}.\ \log|0(1/\lambda_{ni})|=-1/\rho,$$

which shows that $G(s)$ is of order ρ.

Let $E(\theta,\alpha)$ be the complementary set of $\{\tau'\}$, where $F(s;\theta,\alpha)$ has the subset $\{C_{\tau'}\}$ of $\{C_\tau\}$ as its order-curves. Then $E(\theta,\alpha)$ is evidently an open set. Now we can prove

(3. 1) $$E(\theta_1,\alpha)\frown E(\theta_2,\alpha)=0\quad\text{for}\quad\theta_1\neq\theta_2.$$

In fact, if there should exist one curve $C_{\tau''}$ such that

$$\tau''\in E(\theta_1,\alpha)\frown E(\theta_2,\alpha)\neq 0,$$

then by the entirely similar discussion as above, $C_{\tau''}$ would not be the order-curve of $G(s)$. On the other hand, by (2. 2) and an extension of G. Pólya's theorem[6], $G(s)$ has every curve C_τ $(-\infty<\tau<+\infty)$ as its order-curve, which leads us to a contradiction. Thus (3. 1) is proved.

If $E(\theta,\alpha)\neq 0$ holds for $0\leqq\theta\leqq\gamma$ (γ: a fixed constant), by (3. 1), the function-family $\{F(s;\theta,\alpha)\}$ is at most of enumerable power, which contradicts the power of continuum of $\{F(s;\theta,\alpha)\}$. Hence, at least one θ', $E(\theta',\alpha)=0$. In other words, $F(s;\theta',\alpha)$ has every curve C_τ $(-\infty<\tau<+\infty)$ as its order-curves. If $\alpha=(-1)^{1/2}$ $(=1)$, (a) ((b)) of Theorem II holds, q.e.d.

References

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