125. *Notes on Some Theorems on the Sphere*

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Borsuk[2] proved that if f is a continuous mapping of the n-dimensional sphere S^n into the n-dimensional Euclidean space E^n, then f maps some pair of antipodal points into a single point, which had been conjectured by Ulam. This Borsuk-Ulam theorem has been extended by Tucker[4] such that if f is a continuous mapping of S^n into itself with the degree 0, then f maps some pair of antipodal points into a single point. In this note in §1 we shall have an extension of these theorems.

Borsuk[2] proved also that if S^n is covered by $n+1$ closed sets, then at least one of them contains an antipodal pair, which is now called the theorem of Lusternik-Schnirelmann-Borsuk. In §2 we shall have an extension of this theorem and a consequence of this extension.

§1. Now we prove the following:

Theorem 1. *Let f be a continuous mapping of S^n into itself. If f has an even degree, then f maps some pair of antipodal points into a single point.*

Proof. Assume that S^n is the unit sphere in E^{n+1}. Let f be a continuous mapping which satisfies the condition of Theorem. Suppose on the contrary that $f(x) \neq f(x^*)$ for every $x \in S^n$, where x^* is the antipodal point of x. Using vectorial notation, put

$$g(x) = \frac{f(x)-f(x^*)}{|f(x)-f(x^*)|}.$$

Then g is a continuous mapping of S^n into itself. Since

$$g(x^*) = \frac{f(x^*)-f(x)}{|f(x^*)-f(x)|} = -g(x)$$

for every $x \in S^n$, g maps antipodal points of S^n into antipodal points of S^n. Therefore g has an odd degree by a theorem of Borsuk[2].

Now we shall prove that $|f(x)-g(x)|<2$ for every $x \in S^n$. Since from this fact it follows that g will be homotopic to f and that g will have the same degree to that of f (i.e. an even degree), we shall have a contradiction, and the proof of Theorem will be complete.

To prove that $|f(x)-g(x)|<2$ for every $x \in S^n$, suppose on the contrary that there exists a point $p \in S^n$ with $|f(p)-g(p)|=2$. Then we have

$$f(p) = -g(p) = -\frac{f(p)-f(p^*)}{|f(p)-f(p^*)|}.$$

Therefore

$$f(p^*) = (1+|f(p)-f(p^*)|)f(p).$$

Since $|f(p)|=|f(p^*)|=1$, we have $|f(p)-f(p^*)|=0$. Then $f(p)=f(p^*)$, which is a contradiction, and the proof is complete.

§ 2. Now we prove the following:

Theorem 2. *Let $F_i (i=0,1,\dots,n)$ be $n+1$ closed subsets of S^n with $\bigcap_{i=0}^n F_i=0$. Then there exists a point $p \in S^n$ such that $p \in F_i$ if and only if $p^* \in F_i$.*

Proof. Let $a_0, a_1, \dots, a_n$ be linearly independent points in E^n. Put

$$f_i(x) = d(x, F_i) \qquad (i = 0, 1, \dots, n)$$

for every $x \in S^n$. Since $\bigcap_{i=0}^n F_i=0$, for each $x \in S^n$ there exsists an i with $f_i(x) > 0$. Using vectorial notation, put

$$g(x) = \frac{1}{\sum_{i=0}^n f_i(x)} \left(f_0(x)a_0 + f_1(x)a_1 + \cdots + f_n(x)a_n \right).$$

Then g is a continuous mapping of S^n into E^n. By the theorem of Borsuk-Ulam there exists a point $p \in S^n$ with $g(p)=g(p^*)$. Then we have

$$\frac{1}{\sum_{i=0}^n f_i(p)} \left(f_0(p)a_0 + f_1(p)a_1 + \cdots + f_n(p)a_n \right)$$
$$= \frac{1}{\sum_{i=0}^n f_i(p^*)} \left(f_0(p^*)a_0 + f_1(p^*)a_1 + \cdots + f_n(p^*)a_n \right).$$

Therefore

$$\frac{f_i(p)}{\sum_{i=0}^n f_i(p)} = \frac{f_i(p^*)}{\sum_{i=0}^n f_i(p^*)} \qquad (i = 0, 1, \dots, n)$$

and then

$$f_i(p) = c f_i(p^*) \qquad (i = 0, 1, \dots, n),$$

where c is a non-zero constant. It follows that $f_i(p)=0$ if and only if $f_i(p^*)=0$. Then $p \in F_i$ if and only if $p^* \in F_i$, and the proof is complete.

Putting as F_0 the empty set in Theorem 2, we have the following:

Theorem 3. *Let $F_i (i = 1, 2, \dots, n)$ be n closed subsets of S^n. Then there exists a point $p \in S^n$ such that $p \in F_i$ if and only if $p^* \in F_i$.*

Remark. Tucker[4] has proved that the Borsuk-Ulam theorem, his fundamental non-existence theorem, his covering theorems on the sphere etc. form an equivalent system. It is easy to see that our Theorems 2 and 3 also are contained in this equivalent system.

References

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3) S. Lefschetz: Introduction to topology (1949).

4) A. W. Tucker: Some topological properties of disk and sphere, Proc, Cana. Math. Cong., Montreal (1946).