124. On the Existence of Periodic Solutions for Certain Differential Equations

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In this note we shall give the existence theorems on the periodic solutions of the differential equations

$$(1)\qquad \frac{d}{dt}\left(a(x)\frac{dx}{dt}\right)+f(x)\frac{dx}{dt}+g(x)=e(t)$$

$$(2)\qquad a(x)\frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=e(t)$$

where $e(t)$ is a periodic function of t with the least positive period ω and $\int_0^\omega e(t)dt=0$, and $|e(t)|\leqq e$. Moreover, we suppose that $a'(x)$, $g(x)$ and $e(t)$ have continuous derivatives and $f(x)$ is a continuous function.

Of course, the proofs of the following theorems follow from the fixed point theorem. Therefore, it is sufficient to show that the existence of a curve which encloses the domain satisfying the hypotheses of the fixed point theorem.

Theorem 1. *Suppose that the following conditions are satisfied:*

(a) $a(x)>0$ *for all* x.

(b) $\int_0^x f(x)\,dx\,(=F(x))\to\pm\infty$ *as* $x\to\pm\infty$ *respectively.*

(c) *There exists a positive number* x_0 *such that* $x\cdot g(x)\geqq 0$ *for* $|x|\geqq x_0$.

Then the equation (1) *has at least one periodic solution of period* ω.

Proof. We consider a pair of first order equations,

$$(3)\qquad \begin{cases} a(x)\dfrac{dx}{dt}=y-F(x)+E(t)=y-F(x)+\displaystyle\int_0^t e(t)\,dt \\ \dfrac{dy}{dt}=-g(x)\end{cases}$$

instead of the equation (1).

For a positive number ε, we choose an x-value $\xi\,(\geqq x_0)$ such that

$$F(x)>\max_t E(t)+\varepsilon \qquad \text{for } x\geqq\xi,$$
$$F(x)<\min_t E(t)-\varepsilon \qquad \text{for } x\leqq-\xi,$$

and a positive number η such that $\eta\leqq\varepsilon/A(\xi)$ and $\eta\leqq-\varepsilon/A(-\xi)$ where $A(x)=\int_0^x a(x)\,dx$.

Now, we consider three functions

$$\Gamma_1(x,\ y) = \tfrac{1}{2}[y-\eta A(x)]^2 \qquad \text{for} \quad |x| \leqq \xi,$$
$$\Gamma_2(x,\ y) = \tfrac{1}{2}[y-\eta A(\xi)]^2 + \Phi(x) - \Phi(\xi) \qquad \text{for} \quad x \geqq \xi,$$
$$\Gamma_3(x,\ y) = \tfrac{1}{2}[y-\eta A(-\xi)]^2 + \Phi(x) - \Phi(-\xi) \qquad \text{for} \quad x \leqq -\xi,$$

where $\Phi(x) = \int_0^x a(x)\,g(x)\,dx$.

Then we have

$$\frac{d\Gamma_1(x,\ y)}{dt} = [y-\eta A(x)]\left[\frac{dy}{dt} - \eta a(x)\frac{dx}{dt}\right]$$
$$= -\eta[y-\eta A(x)]^2 + [y-\eta A(x)]$$
$$\{-g(x) - \eta[-F(x) + \eta A(x) + E(t)]\}$$
$$\frac{d\Gamma_2(x,\ y)}{dt} = [y-\eta A(\xi)]\frac{dy}{dt} + a(x)\,g(x)\frac{dx}{dt}$$
$$= g(x)[-F(x) + E(t) + \eta A(\xi)]$$
$$\frac{d\Gamma_3(x,\ y)}{dt} = g(x)[-F(x) + E(t) + \eta A(-\xi)].$$

Accordingly, if we choose $|y-\eta A(x)|$ sufficiently large, we have $\frac{d\Gamma_1}{dt} < 0$ for $|x| \leqq \xi$, and $\frac{d\Gamma_i(x,\ y)}{dt} \leqq 0$ ($i=2,\ 3$) is clear in the sense of ξ and η. Hence, we choose $C(>0)$ sufficiently large, and consider three curves

$$\Gamma_1(x,\ y) = C \qquad \text{for} \quad |x| \leqq \xi,$$
$$\Gamma_2(x,\ y) = C \qquad \text{for} \quad x \geqq \xi,$$
$$\Gamma_3(x,\ y) = C \qquad \text{for} \quad x \leqq -\xi.$$

These curves enclose either a bounded domain D (it is the case $\Phi(x) \to \infty$ as $|x| \to \infty$) or an unbounded domain D. In the first case, the curve $(x(t),\ y(t))$ $(t \geqq 0)$ remains in D if $(x(0),\ y(0)) \in D$. In the second case, since y is bounded for $(x,\ y) \in D$, (3) shows that if we take ξ_1 sufficiently large, $\frac{dx}{dt} < 0$ for $x = \xi_1$, $\frac{dx}{dt} > 0$ for $x = -\xi_1$. Then the same as above is true for the domain $(x,\ y) \in D$, $|x| \leqq \xi_1$.

Theorem 2. *The equation* (2) *has at least one periodic solution of period* ω, *if the following conditions are satisfied:*

(a) $a(x) > 0$ *for all* x,
and $x \cdot a'(x) > 0$ *for* $|x| \geqq x_0$.

(b) $F(x)/a(x) \to \pm\infty$ *as* $x \to \pm\infty$ *respectively,*
and $F^2(x) > \dfrac{e^2 \cdot a^2(x)}{4a'(x)[g(x)-e(t)]}$ *for* $|x| \geqq x_0$,
where $F(x) = \int_0^x f(x)dx$.

(c) $x[g(x)-e(t)] > 0$ *for* $|x| \geqq x_0$,

where x_0 *is a positive number.*

Proof. We consider a pair of first order equations,

$$\begin{cases} a(x)\dfrac{dx}{dt}=a(x)y-F(x) \\ a(x)\dfrac{dy}{dt}=-\dfrac{a'(x)}{a(x)}F(x)y+\dfrac{a'(x)}{a^2(x)}F^2(x)-g(x)+e(t) \end{cases}$$

instead of the equation (2).

First we take $\xi(\geqq x_0)$ such that $x\cdot F(x)>0$ for $|x|\geqq\xi$.

From the hypotheses we have

$$(4)\qquad 4\frac{a'(x)}{a^4(x)}F^2(x)[g(x)-e(t)]>0$$

for $x\geqq\xi$, and hence, there exists a continuous function $\bar{\psi}(x)$ which satisfies following inequalities

$$(5)\qquad \psi^2(x)+\left[\frac{e(t)}{a(x)}-\frac{a'(x)}{a^3(x)}F^2(x)\right]\psi(x)+\left[\frac{a'(x)}{a^3(x)}F^2(x)+\frac{e(t)}{a(x)}\right]^2 -4\frac{a'(x)}{a^4(x)}g(x)F^3(x)\leqq 0$$

and
$$\bar{\psi}(x)+\frac{g(x)}{a(x)}\geqq 0.$$

Because, if we denote the ψs which always cancel the left side of the inequality (5), by $\psi_1(x,t)$, $\psi_2(x,t)$ $(\psi_1(x,t)>\psi_2(x,t))$ respectively, then we have

$$\min_t \psi_1(x,t)-\max_t \psi_2(x,t)\geqq -2\frac{e}{a(x)} +4\sqrt{\frac{a'(x)}{a^4(x)}F^2(x)[g(x)-e]}>0,$$

and hence, if we take $\bar{\psi}(x)$ satisfying

$$\min_t \psi_1(x,t)\geqq\bar{\psi}(x)\geqq\max_t \psi_2(x,t),$$

then we have

$$\bar{\psi}(x)+\frac{g(x)}{a(x)}\geqq\frac{e}{a(x)}+\frac{a'(x)}{a^3(x)}F^2(x) -2\sqrt{\frac{a'(x)}{a^4(x)}F^2(x)[g(x)-e]}+\frac{g(x)}{a(x)}\geqq 0.$$

Accordingly, for such a $\bar{\psi}(x)$, we have

$$\frac{d}{dt}\left\{\frac{y^2}{2}+\int_\xi^x\left[\bar{\psi}(x)+\frac{g(x)}{a(x)}\right]dx\right\}=-\frac{a'(x)}{a^2(x)}F(x)y^2 +\left[\bar{\psi}(x)+\frac{a'(x)}{a^3(x)}F^2(x)+\frac{e(t)}{a(x)}\right]y-\frac{F(x)}{a(x)}\left[\bar{\psi}(x)+\frac{g(x)}{a(x)}\right] \leqq 0 \qquad \text{for } x\geqq\xi.$$

For $x\leqq-\xi$, since the inequality (4) is true, and by the condition (b), we can similarly see the existence of a continuous function $\underline{\psi}(x)$ which satisfies the inequality (4) and

$$\underline{\psi}(x)+\frac{g(x)}{a(x)}\leqq 0.$$

In fact, we have

$$\underline{\psi}(x)+\frac{g(x)}{a(x)}\leqq-\frac{e}{a(x)}+\frac{a'(x)}{a^3(x)}F^2(x)$$
$$+2\sqrt{\frac{a'(x)}{a^4(x)}F^2(x)[g(x)-e]}+\frac{g(x)}{a(x)}\leqq 0.$$

Now, take such a continuous function $\underline{\psi}(x)$, then

$$\frac{d}{dt}\left\{\frac{y^2}{2}+\int_{-\xi}^{x}\left[\underline{\psi}(x)+\frac{g(x)}{a(x)}\right]dx\right\}\leqq 0 \qquad \text{for} \quad x\geqq-\xi.$$

Next, we consider three functions

$$\Gamma_1(x, y)=\frac{y^2}{2} \qquad \text{for} \quad |x|\leqq\xi,$$
$$\Gamma_2(x, y)=\frac{y^2}{2}+\int_{\xi}^{x}\left[\bar{\psi}(x)+\frac{g(x)}{a(x)}\right]dx \qquad \text{for} \quad x\geqq\xi,$$
$$\Gamma_3(x, y)=\frac{y^2}{2}+\int_{-\xi}^{x}\left[\underline{\psi}(x)+\frac{g(a)}{a(x)}\right]dx \qquad \text{for} \quad x\leqq-\xi.$$

As we have seen above, $\frac{d}{dt}\Gamma_i(x, y)\leqq 0$ $(i=2, 3)$ for $x\geqq\xi$, $x\leqq-\xi$ respectively. On the other hand, for $\Gamma_1(x, y)$,

$$\frac{d\Gamma_1(x, y)}{dt}=-\frac{a'(x)}{a^2(x)}F(x)y^2+\left[\frac{a'(x)}{a^3(x)}F^2(x)-g(x)+e(t)\right]y,$$

so, if we take $|y|$ sufficiently large, we have $\frac{d}{dt}\Gamma_1(x, y)\leqq 0$. Hence, let C be sufficiently large, and consider three curves, $\Gamma_i(x, y)=C$ $(i=1, 2, 3)$, then if these curves enclose a bounded domain, the theorem is clear from $\frac{d}{dt}\Gamma_i(x, y)\leqq 0$ $(i=1, 2, 3)$. In the case when these three curves enclose an unbounded domain D, since y is bounded for $(x, y)\in D$, $dx/dt=y-\frac{F(x)}{a(x)}$ shows that if we take ξ_1 sufficiently large, $dx/dt<0$ for $x=\xi_1$ and $dx/dt>0$ for $x=-\xi_1$.

Remark. In the above theorem, if two constants α and β exist such that

$$a'(x)\cdot\operatorname{sign} x\geqq\alpha>0 \qquad \text{for} \quad |x|\geqq x_0,$$
$$[g(x)-e(t)]\cdot\operatorname{sign} x\geqq\beta>0 \qquad \text{for} \quad |x|\geqq x_0,$$

then the condition (b) is simplified as follows:

(b′) $$\frac{F(x)}{a(x)}\to\pm\infty \qquad \text{as } x\to\pm\infty \text{ respectively.}$$

In fact, from the existence of α and β, and the condition (b′), it can be easily seen that there exists an x-value $\xi\,(\geqq x_0)$ such that:

$$F^2(x)>\frac{e^2\cdot a^2(x)}{4\,a'(x)[g(x)-e(t)]} \qquad \text{for} \quad |x|\geqq\xi,$$