123. On Spaces Having the Weak Topology with Respect to Closed Coverings

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Let X be a topological space and $\{A_\alpha\}$ a closed covering of X. We shall say that *X has the weak topology with respect to* $\{A_\alpha\}$, if the union of any subcollection $\{A_\beta\}$ of $\{A_\alpha\}$ is closed in X and any subset of $\bigcup_\beta A_\beta$ whose intersection with each A_β is open relative to the subspace topology of A_β is necessarily open in the subspace $\bigcup_\beta A_\beta$; the word "open" may, of course, be replaced by "closed".

According to this definition any CW-complex K in the sense of J. H. C. Whitehead[1] has the weak topology with respect to the closed covering which consists of the closures of all the cells of K. Thus the theorems concerning spaces having the weak topology with respect to closed coverings are applicable to CW-complexes which play an important rôle in algebraic topology.

Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. In this paper we are concerned primarily with the problem: what property of each subspace A_α has influence upon the whole space X? For example, if each A_α consists of a single point (or more generally if each A_α is discrete), X is discrete. It will be shown below that if each subspace A_α is (completely or perfectly) normal, so is X. Our main theorem is that if each subspace A_α is metrizable, then any subset of X is paracompact and perfectly normal. Since the closure of each cell of a CW-complex is a compact metrizable space, it follows immediately from our theorem that any subset of a CW-complex is paracompact and perfectly normal[2].

§ 1. Product Spaces.

Lemma 1. *Let $\{A_\alpha\}$ be a locally finite (=neighbourhood finite in the sense of S. Lefschetz) closed covering of a topological space X. Then X has the weak topology with respect to $\{A_\alpha\}$.*

Lemma 2. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. Then a mapping f of X into*

1) J. H. C. Whitehead, Bull. Amer. Math. Soc., **55**, 213-245 (1949).

2) The paracompactness is proved independently for simplicial complexes with the weak topology by D. G. Bourgin, Proc. Nat. Acad. Sci. U.S.A., **38**, 305-313 (1952); J. Dugundji, Portugaliae Math., **11**, 7-10-b (1952); H. Miyazaki, Tohoku Math. Jour., **4**, 83-92 (1952); K. Morita, Amer. Jour. Math., **75**, 205-223 (1953) and for CW-complexes by H. Miyazaki, Tohoku Math. Jour., **4**, 309-313 (1952).

another topological space Y *is continuous if and only if* $f|A_\alpha$ *is continuous for each* A_α.

Lemma 3. *Let* X *be a topological space having the weak topology with respect to a closed covering* $\{A_\alpha\}$. *If* Y *is a locally compact Hausdorff space, the product space* $X\times Y$ *has the weak topology with respect to the closed covering* $\{A_\alpha\times Y\}$.

Since Lemmas 1 and 2 are obvious, we shall prove only Lemma 3. For this purpose it is sufficient to prove that if G is a subset of $X\times Y$ and its intersection $G\frown(A_\alpha\times Y)$ with each $A_\alpha\times Y$ is open in the subspace $A_\alpha\times Y$, then G is open in $X\times Y$.

Let (x_0, y_0) be any point of G and let us assume that x_0 belongs to A_{α_0}. If we put $H=\{y|(x_0, y)\in G,\ y\in Y\}$, then H is an open set of Y, because $H=\{y|(x_0, y)\in G\frown(A_{\alpha_0}\times Y)\}$ and by the assumption $G\frown(A_{\alpha_0}\times Y)$ is open in $A_{\alpha_0}\times Y$.

Since $y_0\in H$ and Y is a locally compact (=bicompact) Hausdorff space, there is an open set W such that $\overline{W}$ is compact and $y_0\in W$, $\overline{W}\subset H$. If we put $V=\{x|x\times\overline{W}\subset G\}$, we have $V\frown A_\alpha=\{x|x\times\overline{W}\subset G\frown(A_\alpha\times Y)\}$ and, since $G\frown(A_\alpha\times Y)$ is open in $A_\alpha\times Y$ and $\overline{W}$ is compact, $V\frown A_\alpha$ is open in A_α. Hence V is open in X. Since $(x_0, y_0)\in V\times W$, $V\times\overline{W}\subset G$, G is an open set of $X\times Y$. This proves Lemma 3.

Theorem 1. *Let* X *be a topological space having the weak topology with respect to a closed covering* $\{A_\alpha\}$. *If* $\{B_\beta\}$ *is a locally finite closed covering of a locally compact Hausdorff space* Y, *then* $X\times Y$ *has the weak topology with respect to the closed covering* $\{A_\alpha\times B_\beta\}$.

Proof. Let $\{A_\alpha\times B_\beta|\alpha\in\Gamma_\beta,\ \beta\in\Delta\}$ be any subcollection of $\{A_\alpha\times B_\beta\}$ and let Z be its union. Since $Z_\beta=\bigcup\{A_\alpha\times B_\beta|\alpha\in\Gamma_\beta\}$ is closed in $X\times Y$ and $\{X\times B_\beta|\beta\in\Delta\}$ is locally finite in $X\times Y$, Z is a closed set of $X\times Y$. If F is a subset of Z and its intersection with each $A_\alpha\times B_\beta\,(\alpha\in\Gamma_\beta,\ \beta\in\Delta)$ is closed in $A_\alpha\times B_\beta$, then $F\frown Z_\beta$ is closed in Z_β as is shown by applying Lemma 3 to $\bigcup\{A_\alpha|\alpha\in\Gamma_\beta\}$ and B_β. Since $F\frown Z_\beta$ is closed in $X\times B_\beta$ and $F=\bigcup\{F\frown Z_\beta|\beta\in\Delta\}$, F is closed in $X\times Y$ and a fortiori closed in Z. This completes our proof.

§ 2. Normality.

Theorem 2. *Let* X *be a topological space having the weak topology with respect to a closed covering* $\{A_\alpha\}$. *If each* A_α *is normal as a subspace,* X *is a normal space. Furthermore, if* $\dim A_\alpha\leqq n$ *for each* α, *we have* $\dim X\leqq n$.

Proof. We assume that the set of indices α consists of all transfinite ordinals α less than some ordinal η. Thus $X=\bigcup\{A_\alpha|\alpha<\eta\}$. Let us put for $\tau<\eta$

$$P_\tau=\bigcup\{A_\alpha|\alpha\leqq\tau\},\quad Q_\tau=\bigcup\{A_\alpha|\alpha<\tau\}.$$

Let F be any closed set of X and f any continuous map of F into the closed unit interval $I=\{t|0\leqq t\leqq 1\}$. We shall show that f can be extended over X continuously. For this purpose let us assume that for every $\alpha<\tau$ there exists a continuous map $f_\alpha: P_\alpha \cup F \to I$ such that if $\beta<\alpha$ we have $f_\alpha(x)=f_\beta(x)$ for $x \in P_\beta \cup F$, where we put $f_0=f$. Then we define a map $g: Q_\tau \cup F \to I$ by $g(x)=f_\alpha(x)$ for $x \in P_\alpha \cup F$. It is clear that g is single-valued. On the other hand, by Lemma 2 the continuity of g follows from the fact that $g|P_\alpha=f_\alpha|P_\alpha$, $g|F=f$ for $\alpha<\tau$.

Since A_τ is normal, the map $g|L$, where $L=A_\tau \cap (Q_\tau \cup F)$, can be extended to a continuous map $h: A_\tau \to I$. If we define $f_\tau: P_\tau \cup F \to I$ by

$$f_\tau(x) = \begin{cases} g(x) & \text{for } x \in Q_\tau \cup F, \\ h(x) & \text{for } x \in A_\tau, \end{cases}$$

then $f_\tau|P_\alpha \cup F=f_\alpha$ for $\alpha<\tau$, and f_τ is a continuous map.

Thus for any $\tau<\eta$ there can be found, by transfinite induction, a continuous map $f_\tau: P_\tau \cup F \to I$ such that $f_\sigma=f_\tau|P_\sigma \cup F$ for $\sigma<\tau$. Hence by the same method as that of the construction of g from f_α's we see the existence of a continuous map $\varphi: X=\bigcup_\tau P_\tau \cup F \to I$ such that $\varphi|F=f$. This proves the normality of X.

Furthermore, if dim $A_\alpha \leqq n$ for each α, we can follow the above argument with I replaced by an n-sphere S^n. This completes our proof.

Remark. By an extension theorem of C. H. Dowker[3] for collectionwise normal spaces we can prove similarly that for a space X having the weak topology with respect to a closed covering $\{A_\alpha\}$ the collectionwise normality of each A_α implies the collectionwise normality of X.

Theorem 3. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. If each subspace A_α is completely (perfectly) normal, so is X.*

The part concerning the complete normality follows from Theorem 2 and Lemma 4 below, since a space is completly normal if each open subset is normal. The part concerning the perfect normality is proved, in view of Theorem 2, if we show that any open subset of X is an F_σ-set; but the latter is easily verified.

Lemma 4. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. If Z is an open (closed) subset of X, Z has the weak topology with respect to $\{Z \cap A_\alpha\}$.*

§ 3. A Lemma. Before proceeding to our main theorem we find it convenient to prove the following lemma.

Lemma 5. *Let $\{B, C\}$ be a closed covering of a topological space*

3) C. H. Dowker, Arkiv. f. Mat., **2**, 307–713 (1952).

Y. In case C is metrizable, there exists a correspondence ψ which associates with every open set G of B an open set $\psi(G)$ of Y and has the property that $\psi(G)\cap B=G$, and $\psi(G)\cap\psi(H)=0$ if and only if $G\cap H=0$.

Proof. Let us denote by ρ a metric of C which induces the given topology of C. For any open set L of the subspace $B\cap C=C_0$ we define[4]

$$\varphi(L)=\{x\,|\,\rho(y,x)<\frac{1}{2}\rho(y,C_0-L) \text{ for some point } y \text{ of } L\}.$$

Then $\varphi(L)$ is clearly an open set of C and $\varphi(L)\cap C_0=L$. Moreover, if $L_1\cap L_2=0$, then $\varphi(L_1)\cap\varphi(L_2)=0$. Because, if $\varphi(L_1)$ and $\varphi(L_2)$ have a point x in common, there exist two points $y_i\in L_i$, $i=1,2$ such that $\rho(y_i,x)<\frac{1}{2}\rho(y_i, C_0-L_i)$, $i=1,2$. Let us assume that $\rho(y_1, C_0-L_1)\leqq\rho(y_2, C_0-L_2)$. Then we have $\rho(y_2,y_1)\leqq\rho(x,y_1)+\rho(x,y_2)<\rho(y_2,C_0-L_2)$. This shows that $y_1\in L_2$ and hence $y_1\in L_1\cap L_2$, contradicting to the assumption that $L_1\cap L_2=0$.

Now let us put for any open set G of B

$$\psi(G)=G\cup\varphi(G\cap C).$$

Here it is to be noted that $\varphi(G\cap C)$ is an open set of C. Then we have

$$\psi(G)\cap B=G,\qquad \psi(G)\cap C=\varphi(G\cap C),$$

and $\psi(G)$ is an open set of Y by Lemma 1.

Let us assume that $G\cap H=0$, $x\in\psi(G)\cap\psi(H)$, where G, H are open sets of B. Then the point x must belong to C and hence $x\in\varphi(G\cap C)\cap\varphi(H\cap C)$, but this implies $(G\cap C)\cap(H\cap C)\neq 0$ which contradicts the assumption that $G\cap H=0$. This completes the proof.

§ 4. **Paracompactness.** Now we shall prove our main theorem.

Theorem 4. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$, and let each A_α be metrizable as a subspace. Then X is paracompact and normal.*

Proof. Since X is normal by Theorem 2, it is sufficient to prove the paracompactness of X.

We assume that the set of indices α consists of all ordinals α less than a fixed ordinal η and put, for each $\tau<\eta$,

$$P_\tau=\cup\{A_\alpha\,|\,\alpha\leqq\tau\},\quad Q_\tau=\cup\{A_\alpha\,|\,\alpha<\tau\}.$$

Let $\mathfrak{G}$ be any open covering of X. We shall prove the existence of a locally finite refinement $\mathfrak{V}$ of $\mathfrak{G}$. The construction of $\mathfrak{V}$ will be performed by transfinite induction. For this purpose we assume that for each α less than $\tau(<\eta)$ there exist two open coverings

$$\mathfrak{U}_\alpha=\{U(\lambda,\alpha)\,|\,\lambda\in\Omega_\alpha\},\quad \mathfrak{W}_\alpha=\{W_\alpha(x)\,|\,x\in P_\alpha\}$$

4) Cf. W. T. van Est, Fund. Math., **39**, 179-188 (1953).

of P_α with the following properties:

(1_α) $\mathfrak{U}_\alpha$ is a refinement of $\mathfrak{G} \cap P_\alpha = \{G \cap P_\alpha \mid G \in \mathfrak{G}\}$.

(2_α) $\mathfrak{U}_\alpha$ is a locally finite open covering of P_α.

(3_α) In case $\beta < \alpha$ we have $\Omega_\beta \subset \Omega_\alpha$ and

$$U(\lambda, \beta) = U(\lambda, \alpha) \cap P_\beta, \quad \text{for } \lambda \in \Omega_\beta.$$

(4_α) In case $\beta < \alpha$ and $U(\lambda, \beta) \subset G_{\kappa(\lambda)} \in \mathfrak{G}$, we have also

$$U(\lambda, \alpha) \subset G_{\kappa(\lambda)}.$$

(5_α) For any point $x \in P_\alpha$, $x \in W_\alpha(x)$ and

$$\Gamma_\alpha(x) = \{\lambda \mid W_\alpha(x) \cap U(\lambda, \alpha) \neq 0\} \text{ is a finite set.}$$

(6_α) In case $\beta < \alpha$ and $x \in P_\beta$, we have

$$\Gamma_\beta(x) = \Gamma_\alpha(x), \quad W_\beta(x) = W_\alpha(x) \cap P_\beta.$$

Now let us put $\Phi = \bigcup \{\Omega_\alpha \mid \alpha < \tau\}$ and

$$U_*(\lambda) = \bigcup \{U(\lambda, \alpha) \mid \alpha < \tau\}, \text{ for } \lambda \in \Phi,$$

where $U(\lambda, \alpha)$ means the empty set for those α that $\lambda \bar{\in} \Omega_\alpha$. Similarly we put, for any point x of Q_τ,

$$W_*(x) = \bigcup \{W_\alpha(x) \mid \alpha < \tau\}, \quad \Gamma_*(x) = \bigcup \{\Gamma_\alpha(x) \mid \alpha < \tau\},$$

where $W_\alpha(x)$, $\Gamma_\alpha(x)$ mean the empty set for a point x not contained in P_α.

Then we have clearly $U_*(\lambda) \subset G_{\kappa(\lambda)}$ and

$$U_*(\lambda) \cap P_\alpha = U(\lambda, \alpha), \quad W_*(x) \cap P_\alpha = W_\alpha(x).$$

Hence $U_*(\lambda)$, $W_*(x)$ are open sets of Q_τ by the property of the weak topology.

By (6_α) $\Gamma_*(x)$ is a finite set. For $\lambda \in \Phi - \Gamma_*(x)$ we have $W_*(x) \cap U_*(\lambda) = \bigcup_\alpha (W_*(x) \cap U(\lambda, \alpha)) = \bigcup_\alpha (W_*(x) \cap U(\lambda, \alpha) \cap P_\alpha)$ $= \bigcup_\alpha (W_\alpha(x) \cap U(\lambda, \alpha)) = 0$. Therefore $\{U_*(\lambda) \mid \lambda \in \Phi\}$ is a locally finite open covering of Q_τ and it is a refinement of $\mathfrak{G} \cap Q_\tau$.

Now we apply Lemma 5 to the space P_τ and the closed covering $\{Q_\tau, A_\tau\}$ of P_τ. Using the same notation ψ as in Lemma 5 we put $L = \bigcup \{\psi(W_*(x)) \mid x \in Q_\tau\}$. Then L is open in P_τ and $Q_\tau \subset L$. Since P_τ is normal, there exists an open set M of P_τ such that $Q_\tau \subset M$, $\bar{M} = \bar{M} \cap P_\tau \subset L$.

For $\lambda \in \Phi$ let us put

$$U(\lambda, \tau) = \psi(U_*(\lambda)) \cap M \cap G_{\kappa(\lambda)}.$$

Then $U(\lambda, \tau)$ is an open set of P_τ and is contained in $G_{\kappa(\lambda)}$, and moreover we have $U(\lambda, \tau) \cap Q_\tau = U_*(\lambda)$, since $\psi(U_*(\lambda)) \cap M \cap G_{\kappa(\lambda)} \cap Q_\tau$ $= U_*(\lambda) \cap M \cap G_{\kappa(\lambda)} = U_*(\lambda)$.

As has been proved above, for $\lambda \in \Phi - \Gamma_*(x)$ we have $W_*(x) \cap U_*(\lambda) = 0$ and hence $\psi(W_*(x)) \cap \psi(U_*(\lambda)) = 0$ by the property of ψ. Therefore $\{U(\lambda, \tau) \mid \lambda \in \Phi\}$ is locally finite in L. It is also locally finite in $P_\tau -$, $\bar{M}$ since $U(\lambda, \tau) \subset M$. Thus $\{U(\lambda, \tau) \mid \lambda \in \Phi\}$ is locally finite in P_τ, since L and $P_\tau - \bar{M}$ are open in P_τ and $P_\tau = L \cup (P_\tau - \bar{M})$.

Let us put further $C = P_\tau - \bigcup \{U(\lambda, \tau) \mid \lambda \in \Phi\}$. Then C is closed in P_τ and $C \subset A_\tau - Q_\tau$. Since P_τ is normal there exists an open set N of P_τ such that $Q_\tau \subset N$, $\bar{N} \cap C = 0$.

By the assumption of the theorem A_τ is a metrizable space and hence C is paracompact[5]. Therefore there exists a locally finite closed covering $\{F(\mu) \mid \mu \in \Psi\}$ of C which is a refinement of $\mathfrak{G} \cap C$. Because of the paracompactness of A_τ there is a locally finite system $\{U(\mu, \tau) \mid \mu \in \Psi\}$ of open sets of A_τ such that it is a refinement of $\mathfrak{G} \cap A_\tau$ and $F(\mu) \subset U(\mu, \tau) \subset A_\tau - \bar{N}$.[6]

Thus we have $C \subset \bigcup \{U(\mu, \tau) \mid \mu \in \Psi\} \subset A_\tau - \bar{N} = P_\tau - \bar{N}$. Therefore $U(\mu, \tau)$ is open in $P_\tau - \bar{N}$ and hence in P_τ.

Since $\{U(\mu, \tau) \mid \mu \in \Psi\}$ is locally finite both in N and in $A_\tau - Q_\tau$ $(= P_\tau - Q_\tau)$ and N, $P_\tau - Q_\tau$ are open in P_τ, $\{U(\mu, \tau) \mid \mu \in \bar{\Psi}\}$ is locally finite in P_τ $(= N \cup (P_\tau - Q_\tau))$.

Let us put, for a point x of Q_τ,

$$W_\tau(x) = \psi(W_*(x)) \cap N.$$

Then $W_\tau(x)$ is an open set of P_τ and we have

$$W_\tau(x) \cap Q_\tau = W_*(x), \quad W_\tau(x) \cap U(\mu, \tau) \subset N \cap (P_\tau - N) = 0, \text{ for } \mu \in \Psi.$$

If we denote the union of Φ and $\bar{\Psi}$ by Ω_τ, we have $\Gamma_\tau(x) = \{\lambda \mid W_\tau(x) \cap U(\lambda, \tau) \neq 0,\ \lambda \in \Omega_\tau\} = \{\lambda \mid W_\tau(x) \cap U(\lambda, \tau) \neq 0,\ \lambda \in \Phi\} = \{\lambda \mid \psi(W_*(x)) \cap \psi(U_*(\lambda)) \neq 0,\ \lambda \in \Phi\} = \{\lambda \mid W_*(x) \cap U_*(\lambda) \neq 0,\ \lambda \in \Phi\} = \Gamma_*(x)$.

For a point x of $P_\tau - Q_\tau$ there exists an open set $W_\tau(x)$ of P_τ such that $x \in W_\tau(x)$, $W_\tau(x) \subset P_\tau - Q_\tau$ and

$$\Gamma_\tau(x) = \{\lambda \mid W_\tau(x) \cap U(\lambda, \tau) \neq 0,\ \lambda \in \Omega_\tau\}$$

is a finite set.

Let us put

$$\mathfrak{U}_\tau = \{U(\lambda, \tau) \mid \lambda \in \Omega_\tau\}, \quad \mathfrak{W}_\tau = \{W_\tau(x) \mid x \in P_\tau\}.$$

Then these coverings satisfy the conditions (1_τ) to (6_τ).

Thus by transfinite induction we can find open coverings $\mathfrak{U}_\alpha$, $\mathfrak{W}_\alpha$ satisfying the conditions (1_α) to (6_α) for each $\alpha < \eta$.

Let us put finally

$$\mathfrak{V} = \{V(\lambda) \mid \lambda \in \Omega\}$$

where

$$V(\lambda) = \bigcup \{U(\lambda, \alpha) \mid \alpha < \eta\}, \quad \Omega = \bigcup \{\Omega_\alpha \mid \alpha < \eta\},$$

and $U(\lambda, \alpha)$ means the empty set for $\lambda \bar{\in} \Omega_\alpha$. By the same arguments as those for coverings $\{U_*(\lambda)\}$, $\{W_*(x)\}$ described above we can prove that $\mathfrak{V}$ is a locally finite open covering of $X (= \bigcup \{P_\alpha \mid \alpha < \eta\})$ and $\mathfrak{V}$ is a refinement of $\mathfrak{G}$. Thus the theorem is completely proved.

This theorem can be somewhat sharpened by Theorem 3 and a theorem of C. H. Dowker[7].

Theorem 5. *Under the same assumptions as in Theorem 4,*

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5) Cf. A. H. Stone, Bull. Amer. Math. Soc., **54**, 977-982 (1948).

6) Cf. K. Morita, Jour. Math. Soc. Japan, **2**, 16-33 (1950).

7) C. H. Dowker, Duke Math. Jour., **14** (1947).

any subset of X is paracompact and perfectly normal.

The paracompactness of any subset of X is deduced also readily from Theorem 4 and Lemma 4.

Remark. In general, the word "paracompact" in the conclusion of Theorem 4 cannot be replaced by "metrizable"; but in case $\{A_\alpha\}$ is a locally finite covering one can prove the metrizability of X, since in this case X is a paracompact and locally metrizable space[8], and such a space is easily shown to be metrizable.

8) Because a space which is a finite sum of closed metrizable subspaces is metrizable. Cf. R. H. Bing, Duke Math. Jour., **14**, 511-519 (1947); F. Hausdorff, Fund. Math., **30** (1938). Of course a proof appealing to the known metrizability conditions is possible.