14. *Two Remarks on Dimension Theory for Metric Spaces*

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The purpose of this brief note is to make slight remarks on extensions of the well-known theorems in dimension theory for metric spaces.

First, we can extend Eilenberg-Otto's theorem to the countable dimensional case as follows.

Proposition 1. *A metric space R is countable-dimensional, i.e. it is represented as a countable sum of 0-dimensional spaces if and only if for every collections $\{U_i \mid i=1,2,\cdots\}$ of open sets and $\{F_i \mid i=1,2,\cdots\}$ of closed sets satisfying $F_i \subset U_i$, $i=1,2,\cdots$, there exists a collection $\mathfrak{V}=\{V_i \mid i=1,2,\cdots\}$ of open sets such that*

(1) $F_i \subset V_i \subset U_i, \quad i=1,2,\cdots$

(2) *$\{B(V) \mid V \in \mathfrak{V}\}$ is point-finite, i.e. its order is finite at every point p of R, where $B(V)$ denotes the boundary of V.*

Proof. Since the "only if" part is a direct consequence of [1, Theorem 2], we show only the "if" part. By R. H. Bing's theorem [2] we can find a σ-discrete basis $\mathfrak{U}=\bigcup_{i=1}^{\infty}\mathfrak{U}_i$ for the metric space R. Let $\mathfrak{U}_i=\{U_\gamma \mid \gamma \in \Gamma_i\}$, $U_\gamma=\bigcup_{j=1}^{\infty}F_{\gamma j}$ for closed sets $F_{\gamma j}$. Furthermore, let $U_i=\bigcup\{U_\gamma \mid \gamma \in \Gamma_i\}$, $F_{ij}=\bigcup\{F_{\gamma j} \mid \gamma \in \Gamma_i\}$. Then, since $F_{ij} \subset U_i$, $i, j=1,2,\cdots$, we can find a collection $\mathfrak{V}=\{V_{ij} \mid i,j=1,2,\cdots\}$ of open sets such that $F_{ij} \subset V_{ij} \subset U_i$, $\{B(V) \mid V \in \mathfrak{V}\}$ is point-finite. Letting $V_{ij} \cap U_\gamma = W_{\gamma j}$, $\gamma \in \Gamma_i$ we get a locally finite collection $\mathfrak{W}_{ij}=\{W_{\gamma j} \mid \gamma \in \Gamma_i\}$. Now $\mathfrak{W}=\bigcup\{\mathfrak{W}_{ij} \mid i,j=1,2,\cdots\}$ is a σ-locally finite basis of R such that $\{B(W) \mid W \in \mathfrak{W}\}$ is point-finite. Hence by [1, Theorem 1], we can conclude that R is countable-dimensional.

Next, we can give an extension to the sum-theorem as follows.

Proposition 2. *Let $\{F_\alpha \mid \alpha<\tau\}$ be a covering of a metric space R consisting of subsets F_α with $\dim F_\alpha \leqq n$, $\alpha<\tau$ such that $\{F_\alpha \mid \alpha<\beta\}$ is closed for every $\beta<\tau$. Then $\dim R \leqq n$.*

Proof. E. Michael gave a simple proof of this theorem by use of the sum-theorem for countably many closed sets and locally finite collection of closed sets which is due to K. Morita [3] and partly to M. Katětov [4] and the others. Now, however, let us give a sketch of a direct proof. We assume $F_\alpha \cap F_\beta=\phi$ for every α, β with $\alpha \neq \beta$ without loss of generality.

In the case of $n=0$, let G and H be disjoint closed sets of R. Then we can define, by induction with respect to α,

(1) an open closed set U_α of the subspace F_α such that

$$\overline{[\cup\{S_{\varepsilon(x)/4}(x) \mid x \in \bigcup_{\beta<\alpha} U_\beta\} \cup G]} \cap F_\alpha \subset U_\alpha \subset F_\alpha - \overline{[\cup\{S_{\varepsilon(x)/4}(x) \mid x \in \bigcup_{\beta<\alpha}(F_\beta - U_\beta)\} \cup H]}$$

(2) $\varepsilon(x) > 0$ for each point x of F_α such that

$$S_{\varepsilon(x)}(x) \cap [H \cup (F_\alpha - U_\alpha) \cup (\bigcup_{\beta<\alpha} W_\beta)] = \phi \quad \text{if } x \in U_\alpha,$$

$$S_{\varepsilon(x)}(x) \cap [G \cup U_\alpha \cup (\bigcup_{\beta<\alpha} V_\beta)] = \phi \quad \text{if } x \in F_\alpha - U_\alpha,$$

where we let

$$V_\beta = \cup\{S_{\varepsilon(x)/4}(x) \mid x \in U_\beta\},$$

$$W_\beta = \cup\{S_{\varepsilon(x)/4}(x) \mid x \in F_\beta - U_\beta\}.$$

The proof is easy and left to the reader. Letting

$$V = \cup\{S_{\varepsilon(x)/4}(x) \mid x \in \bigcup_{\alpha<\tau} U_\alpha\},$$

$$W = \cup\{S_{\varepsilon(x)/4}(x) \mid x \in \bigcup_{\alpha<\tau}(F_\alpha - U_\alpha)\},$$

we get open sets V and W satisfying $V \supset G$, $W \supset H$, $V \cap W = \phi$, $V \cup W = R$, which means $\dim R \leqq 0$.

Assume that this proposition has been established for every non-negative integer less than n, then we can decompose every F_α, by the decomposition theorem which is originally due to [4, 5] and can be also deduced from the above inductive assumption in this proof, as follows: $F_\alpha = \bigcup_{i=1}^{n+1} A_{\alpha i}$, $\dim A_{\alpha i} \leqq 0$. Set $A_i = \bigcup_{\alpha<\tau} A_{\alpha i}$; then $\bigcup_{\alpha<\beta} A_{\alpha i}$ is closed in A_i for every $\beta < \tau$. Hence $\dim A_i \leqq 0$, $i = 1, \cdots, n+1$ from the inductive assumption, and hence $\dim R = \dim \bigcup_{i=1}^{n+1} A_i \leqq n$.

References

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