172. On the Uniqueness of the Cauchy Problem for Semi-elliptic Partial Differential Equations. II

By Akira TSUTSUMI
University of Osaka Prefecture

(Comm. by Kinjirô KUNUGI, M.J.A., Dec. 12, 1963)

5. **Proof of Theorem 1.** At first we note that once (2.1), (4.1), and (4) in Lemma 2 are established Theorem 1 can be proved by arguments parallel with them in [1]. Let $V(N^0)$ and $U(0)$ be neighborhoods of N^0 and $x=0$ respectively in which both (2.1) and (4.1) are verified. For any $N_g \in V(N^0)$ and $x_g \in U(0)$ fixed, and $u_g \in C_0^\infty(U(0))$, multiplying $\hat{u}_g(\xi+i\tau N_g)$, which is a translation of Fourier transform of u_g, the both sides of (2.1) and (4.1), and applying Parseval formula, we obtain

$$(5.1)\qquad \sum_{j=1}^{n}\sum_{(1-\frac{1}{m_j})}\int |D^\alpha u_g|^2 \exp(2\tau\langle x, N_g\rangle)\,dx \leq C\int\left\{\sum_{j=1}^{n}|P_0^{(j)}(x_g, D)u_g|^2+|u_g|^2\exp(2\tau\langle x, N_g\rangle)\right\}dx$$

$$(5.2)\qquad \sum_{(1)}\int |D^\alpha u_g|^2\exp(2\tau\langle x, N_g\rangle)\,dx \leq D\int\left\{|P_0(x_g, D)u_g|^2 + |\tau|^2|N_g|^2|P_0^{(1)}(x_g, D)u_g|^2\exp(2\tau\langle x, N_g\rangle)\right\}dx,^{*)}$$

where $\langle x, N_g\rangle$ is $\sum_{j=1}^{n} x_j N_{gj}$.

To replace the weight function $\exp(\langle x, N_g\rangle)$ by $\exp(\varphi_\delta(x))$ in the above, we use a partition of unity designed by Hörmander so that in each corresponding neighborhood $\varphi_\delta(x)$ is almost equal to a linear function. That is:

$$\omega(x)\in C_0^\infty(x; \forall i, |x_i|<1),\ \omega(x)\neq 0 \text{ on } \left(x; \forall i, |x_i|\leqq\frac{1}{2}\right)$$

$g=(g_1, g_2,\cdots, g_n)$; g_i's vary in all integers,

$$\theta(x)=\frac{\omega(x)}{\sum_g \omega(x-g)},\quad \theta_g(x)=\theta(x_1-g_1, x_2-g_2,\cdots, x_n-g_n),$$

and for $u\in C_0^\infty(\Omega)$, $u(x)=\sum_g \theta_g(x)u(x)$.

On a support of $\theta_g(x)$, $\varphi_\delta(x)\leq\varphi_\delta(x_g)+\langle x-x_g, N_g\rangle\leq\varphi_\delta(x)+n\tau^{-1}$ holds where N_g equals to grad $\varphi_\delta(x_g)$. Then for $\tau>\frac{1}{2}$, and $C_1=\exp(2n)C$, $D_1=\exp(2n)D$, we get

*) So far as we can avoid confusion, we use the same letters D, C, etc. for other constants.

(5.3) $$\sum_{j=1}^{n}\sum_{\left(1-\frac{1}{m_j}\right)}\int|D^\alpha u_g|^2\exp(2\tau\varphi_\delta(x))\,dx$$
$$=C_1\int\left\{\left(\sum_{j=1}^{n}|P_0^{(j)}(x_g, D)u_g|^2+|u_g|^2\right)\right\}\exp(2\tau\varphi_\delta(x))\,dx,$$

and we denote the corresponding inequality to (5.2) by (5.4). We note here the 2nd term in the right of (5.3) can be transfered to the left by (4.9) by choosing τ large properly. Now we choose a neighborhood $U_\delta(0)=\left\{x:|x|<\dfrac{\delta}{2}\right\}$ to satisfy the followings by taking a small δ, (a) $\delta<1$, (b) $U_\delta=U_\delta(0)\subset U(0)$, (c) if $x\in U_\delta$, grad $\varphi_\delta(x)\in V(N^0)$, (d) $|$grad $\varphi_\delta(x)-$grad $\varphi_\delta(0)|<\delta$. From I (2) and (a), for x in the support of $\theta_g(x)$ we get

(5.5) $$|(P_0(x, D)-P_0(x_g, D))u_g|^2\leq C\sum_{(1)}|D^\alpha u_g|^2,$$

and for $P_0^{(1)}(x, D)$ we get one, of which the right is replaced $\sum_{(1)}$ by $\sum_{\left(1-\frac{1}{m_1}\right)}$, and call it (5.6). From (d) we get

$$|N_g|=|\text{grad }\varphi_\delta(x_g)|<3\delta.$$

By this and (5.6), (5.5), we can derive the nexts from (5.3) and (5.4) by choosing $\tau\delta$ so large that $C_1\tau\delta>2$, $D_1\tau\delta>2$, hold.

(5.7) $$\sum_{j=1}^{n}\sum_{\left(1-\frac{1}{m_j}\right)}\int|D^\alpha u_g|^2\exp(2\tau\varphi_\delta(x))\,dx$$
$$\leq C_2\int\sum_{j=1}^{n}|P_0^{(j)}(x, D)u_g|^2\exp(2\tau\varphi_\delta(x))\,dx.$$

(5.8) $$\sum_{(1)}\int|D^\alpha u_g|^2\exp(2\tau\varphi_\delta(x))\,dx\leq D_2\int\Big\{|P_0(x, D)u_g|^2$$
$$+(\tau\delta)^2|P_0^{(1)}(x, D)u_g|^2+(\tau\delta)\sum_{\left(1-\frac{1}{m_j}\right)}|D^\alpha u_g|^2\Big\}\exp(2\tau\varphi_\delta(x))\,dx.$$

In what follows $P_0(x, D)$, $P_0^{(j)}(x, D)$ and $\exp(2\tau\varphi_\delta(x))$ are shortened to P_0, $P_0^{(j)}$ and $E(2\tau\varphi_\delta)$ respectively. To sum up with respect to g in the left of (5.7) and (5.8), we can use the estimate $|D^\alpha u(x)|^2\leq 2^{n+1}\sum_g|D^\alpha u_g(x)|^2$. In the right we shall apply (4) of Lemma 2 for them. Setting $t_1=\sqrt{2\tau}$, $t_2=\cdots=t_n=\sqrt{2\tau\delta}$ in $T_s(u, u)$ of Lemma 2 we denote $T_s(u, u)$ by $A_s(u, u)$, $\sum_{s\leq k}A_s(u, u)$ by $B_k(u, u)$ and $T(P_0u, P_0u)$ by A. By Leibniz formula we get

(5.9) $$P_0^{(\alpha)}u_g=\sum_\beta P^{(\alpha+\beta)}u\,\delta^{\frac{\beta^*}{2}}\tau^{\frac{\beta}{2}}\frac{D^\beta\theta}{\beta!},$$

where β^* is $(0, \beta_2, \cdots, \beta_n)$, and setting $\alpha=0$ and using (4) of Lemma 2, we get for a constant C

(5.10) $$\int\sum_g|P_0u_g|^2E(2\tau\varphi_\delta)\,dx\leq C\left\{A+A_1^{\frac{1}{2}}\left(\sum_{j=1}^{n}A_{1-\frac{1}{m_j}}\right)^{\frac{1}{2}}\right\}.$$

For $\alpha=(0, 0,\cdots, 1, 0,\cdots, 0)$ we get
$$_{j}$$

(5.11) $$\tau\int\sum_g|P_0^{(1)}u_g|^2E(2\tau\varphi_\delta)\,dx\leq C\left\{A+A_1^{\frac{1}{2}}\left(\sum_{j=1}^{n}A_{1-\frac{1}{m_j}}\right)^{\frac{1}{2}}\right\},\quad j=1$$

(5.12) $\tau\delta\int\sum_g|P_0^{(j)}u_g|^2E(2\tau\varphi_\delta)\,dx\le C\left\{A+A_1^{\frac12}\left(\sum_{j=1}^n A_{1-\frac{1}{m_j}}\right)^{\frac12}\right\},\quad j\neq 1.$

Thus we get

$$A_1\le C(1+\delta^2\tau)\left\{A+A_1^{\frac12}\left(\sum_{j=1}^n A_{1-\frac{1}{m_j}}\right)^{\frac12}\right\} \tag{5.13}$$

and

$$\sum_{j=1}^n A_{1-\frac{1}{m_j}}\le D(\tau\delta)^{-1}\left\{A+A_1^{\frac12}\left(\sum_{j=1}^n A_{1-\frac{1}{m_j}}\right)^{\frac12}\right\}. \tag{5.14}$$

On the other hand we can easily obtained

$$\tau(1+\delta^2\tau)A_{1-\frac{1}{m_j}}\le CA_1 \tag{5.15}$$

for $u\in C_0^\infty(U_\delta(0))$ which is due to Lemma of [1].

From above three estimates we can derive

$$A_1\le C(1+\delta^2\tau)A \tag{5.16}$$

if $\delta<\delta_0$ and $\tau\delta>M$ are satisfied for constants δ_0 and $M\geqq 1$. This derivation is almost same as that in [1], so we omit here. For any α such that $|\alpha:m|<1$ is satisfied, $|\alpha:m|\le 1-\dfrac{1}{m_j}$ is also satisfied with some j. Hence there is a multi-integer $\rho\ge 0$ such that $|\alpha:m|=1-\dfrac{1}{m_j}-|\rho:m|$ is satisfied. Then by repeated application of (5.15) we get for $u\in C_0^\infty(U_\delta(0))$

$$\tau^{|\rho|}(1+\delta^2\tau)^{|\rho|}\int|D^\alpha u|^2E(2\tau\varphi_\delta)\,dx\le C\sum_{j=1}^n A_{1-\frac{1}{m_j}}. \tag{5.17}$$

Combining (5.15), (5.16), (5.17), and $m_0\left(1-\dfrac{1}{m_1}-|\alpha:m|\right)\le|\rho|$, we get $\{(1+\delta^2\tau)\tau\}^{m_0\left(1-\frac{1}{m_1}-|\alpha:m|\right)}\int|D^\alpha u|^2E(2\tau\varphi_\delta)dx\le CA$ for $u\in C_0^\infty(U_\delta(0))$.

Considering the form of the lower order terms $Q(x,D)$ of $P(x,D)$, we can $P_0(x,D)$ in A by $P(x,D)$ by taking τ large properly. From these (3.1) of Theorem 1 is immediately obtained.

Theorem 2 can be derived from (3.1) of Theorem 1 by usual arguments. And other theorems are obtained similarly to [1].

References

[1] L. Hörmander: On the uniqueness of the Cauchy problem II, Math. Scand., **7**, 177–190 (1959).

[2] ——: Linear Partial Differential Operators, Springer, Berlin (1963).

[3] H. Kumanogo: On the uniqueness for the solution of the Cauchy problem, Osaka Math. Jour. (1963).

[4] S. Mizohata: Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques, Mem. Coll. Sci. Kyoto Univ., ser. A, **31**, 219–239 (1958).

[5] M. H. Protter: Properties of solutions of parabolic equations and inequalities, Canad. Jour. Math., **13**, 331–345 (1961).

[6] F. Trèves: Relations de domination entre opérateurs différentiels, Acta. Math., **101**, 1–139 (1959).
[7] L. Nirenberg: Uniqueness in Cauchy problems for differential equations with constant leading coefficients, Comm. Pure and Appl. Math., **10**, 89–105 (1957).