171. On the Uniqueness of the Cauchy Problem for Semi-elliptic Partial Differential Equations. I

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1. Introduction. In this note we shall prove the inequalities of Carleman type from which we can derive the uniqueness of the Cauchy problem with data on a noncharacteristic surface, having a restriction on its curvature, for some class of semi-elliptic equations. For parabolic equations which are typical in semi-elliptic equations; $\left(\frac{\partial}{\partial t}-L\right)u=0$ (L: 2nd order elliptic operator) M. H. Protter proved the uniqueness when data are given on a time-like surface, (see [5]), S. Mizohata proved it when data are given on any hyperplane not orthogonal to t-axis, (see [4]), and H. Kumanogo generalized the result of Mizohata (see [3]). For elliptic equations which are also typical in semi-elliptic, L. Hörmander proved the uniqueness under mild assumptions. (See [1].)

On the other hand L. Hörmander showed that for any integer $r\geqq 1$ there are examples of non-uniqueness; $\left\{\left(\frac{1}{i}\frac{\partial}{\partial x_2}\right)^r+a(x_1, x_2)\frac{\partial}{\partial x_1}\right\}u=0$, $a(x_1, x_2)=0$ for $x_2\leq 0$. These have several means, but at a point of view of the type of equations these are not semi-elliptic at the origin. (See [2].) This is our motive to study the uniqueness for semi-elliptic equations of higher order. Main tools of our proof are the partition of unity of Hörmander and the inequality of Trèves which is extended for our operators. (See [1], [6].)

2. Notations and some class of semi-elliptic operators. $x=(x_1, x_2, \cdots, x_n)$ is a variable point of n-dimensional euclidean space R^n, and $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ is a vector of Ξ^n dual to R^n, and $\tilde{\xi}$ denotes a vector $(\xi_2, \xi_3, \cdots, \xi_n)$. m is a vector $(m_1, m_2, \cdots, m_n)$ where m_j's are positive integers, α is a vector $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ where α_j's are non-negative integers, by $|\alpha:m|$ we denote $\sum_{j=1}^n \alpha_j/m_j$, $|\alpha|$ is a length of α; $\sum_{j=1}^n \alpha_j$, and m_0 is the minimum of m_j. ξ^α is $\xi_1^{\alpha_1}, \xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$. A polynomial of ξ whose coefficients are functions of x can be written in the following form.

$$P(x, \xi)=P_0(x, \xi)+Q(x, \xi),$$
$$P_0(x, \xi)=\sum_{|\alpha:m|=1} a_\alpha(x)\xi^\alpha,\quad Q(x, \xi)=\sum_{j=1}^n \sum_{|\alpha:m|\leq 1-\frac{1}{m_j}} a_\alpha(x)\xi^\alpha.$$

By $P^{(\alpha)}(x,\xi)$ we denote $\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}P(x,\xi)$ and by $P^{(j)}(x,\xi)$, $\frac{\partial}{\partial\xi_j}P(x,\xi)$. In what follows $\sum_{|\alpha:m|=s}$ is shortened to $\sum_{(s)}$. Substituting ξ_j in $P(x,\xi)$ for $\frac{1}{i}\frac{\partial}{\partial x_j}=D_j$ we obtain a partial differential operator $P(x,D)$. Here we impose on $P(x,D)$ the following conditions:

I. (1) $m_1\geqq m_j$. (2) The coefficients of $P_0(x,D)$ are in $C^{2|m|}(\Omega)$ and those of $Q(x,D)$ are in $C(\Omega)$ and bounded on $\bar{\Omega}$, where Ω is a domain containing $x=0$. (3) For $\alpha=(m_1,0,0,\cdots,0)$, $a_\alpha(0)\neq 0$.

II. $P_0(x,D)$ is *semi-elliptic* at $x=0$, i.e. for any non-zero real vector $P_0(0,\xi)$ does not vanish.

III. Let $\zeta_1=\zeta_1(\tilde{\xi})$ be a root of $P_0(0,\zeta_1,\tilde{\xi})=0$, then $P_0^{(1)}(0,\zeta_1,\tilde{\xi})$ does not vanish for any real $\tilde{\xi}\neq 0$.

IV. Let be $N^0=(-1,0,\cdots,0)$, $N=(N_1,N_2,\cdots,N_n)$ where N_j's are real, and $\xi+i\tau N=(\xi_1+i\tau N_1,\cdots,\xi_n+i\tau N_n)$ where τ is real number. For $m_0\geq 2$ there are neighborhoods $U_0(0)$ of $x=0$, $V_0(N^0)$ of N^0, and constant C_0 such that

$$\sum_{j=1}^n\sum_{\left(1-\frac{1}{m_j}\right)}|(\xi+i\tau N)^\alpha|^2\leq C_0\left\{\sum_{j=1}^n|P_0^{(j)}(x,\xi+i\tau N)|^2+1\right\} \tag{2.1}$$

holds for any $x\in U_0(0)$, any $N\in V_0(N^0)$ and any $(\xi,\tau)\in\Xi^n\times R^1$, $\tau\geqq 1$.

We note that when all m_j are equal II shows $P_0(0,D)$ is elliptic and that IV is derived from I, II, III which are Hörmander's conditions (see [1]). In our case we don't know whether IV is derived from the others or not and' is replaced by a condition for $x=0$ or for a compact set of (ξ,τ), or not. For the case of the constant coefficients and two independent variables L. Nirenberg treated these forms of operators under milder assumptions. (See Theorem 9 of [7].)

3. Theorems.

Theorem 1. Suppose that I, II, III, and IV hold. Then there exist constants $C,\delta_0>0$, $M\geqq 1$, and for any real number τ, δ satisfying $\delta<\delta_0$, $\tau\delta>M$,

$$\sum_{|\alpha:m|\leq 1}\{(1+\tau\delta^2)\tau\}^{m_0\left(1-\frac{1}{m_1}-|\alpha:m|\right)}\tau\int|D^\alpha u|^2\exp(2\tau\varphi_\delta(x))dx \tag{3.1}$$
$$\leq C\int|P(x,D)u|^2\exp(2\tau\varphi_\delta(x))dx$$

holds if $u\in C_0^\infty(U_\delta(0))$, where $\varphi_\delta(x)$ is $(x_1-\delta)^2+\delta\sum_{j=2}^n x_j^2$ and $U_\delta(0)$ is a neighborhood depending on δ.

Theorem 2. Let be $\mathcal{D}=\{x: x_1<x_2^2+x_3^2+\cdots+x_n^2\}$. $P(x,D)$ satisfies the conditions of Theorem 1. Suppose $u\in C^{m_1}$ and satisfies in a neighborhood U_1 of $x=0$ the inequality

$$|P(x,D)u|\leq K\sum_{j=1}^n\sum_{|\alpha:m|\leq 1-\frac{1}{m_j}}|D^\alpha u| \tag{3.2}$$

and $u=0$ for $x\in\mathcal{D}\cap U_1$, then there exists a neighborhood U of $x=0$ such that $u\equiv 0$ in U.

Theorem 3. $P(x, D)$ and $\tilde{P}(x, D)$ satisfy the conditions of Theorem 1 for $m=(m_1,\cdots, m_n)$ and $\tilde{m}=(\tilde{m}_1,\cdots,\tilde{m}_n)$ and furthermore coefficients of $\tilde{P}(x, D)$ are in $C^{|m_1|}(\Omega)$. Let be $\boldsymbol{P}(x, D)=P(x, D)\cdot\tilde{P}(x, D)$. Then there exist constants $\delta_0>0$, $C>0$, $M\geq 1$, for any τ, δ satisfying $\delta<\delta_0$, $\tau\delta>M$

$$(3.3)\quad \sum_{|\alpha:m|\leq 1}\sum_{|\tilde{\alpha}:\tilde{m}|\leq 1}(1+\delta^2\tau)^{m_0\left(1-\frac{1}{m_1}-|\alpha:m|\right)+\tilde{m}_0\left(1-\frac{1}{\tilde{m}_1}-|\tilde{\alpha}:\tilde{m}|\right)}\tau^2\left\{\left(1-C'\frac{1}{\tau}\right)\right\}$$
$$\int|D^\alpha D^{\tilde{\alpha}}u|^2\exp(2\tau\varphi_\delta(z))dx\leq C\int|\boldsymbol{P}(x, D)u|^2\exp(2\tau\varphi_\delta(x))\,dx$$

holds for $u\in C_0(U_\delta(0))$.

Theorem 4. The similar conclusion as in Theorem 2 for the operator $\boldsymbol{P}(x, D)$ if we replace (3.2) for

$$(3.4)\qquad |\boldsymbol{P}(x, D)u|\leq K\sum_{j=1}^{n}\sum_{|\alpha:m|\leq 1-\frac{1}{m_j}}\sum_{|\tilde{\alpha}:\tilde{m}|\leq 1-\frac{1}{m_j}}|D^\alpha D^{\tilde{\alpha}}u|,\ u\in C^{|m_1|+|\tilde{m}_1|}.$$

4. Lemmas which are fundamental.

Lemma 1. Suppose I, II, and III hold, then there exist neighborhoods $U_1(0)$ of $x=0$, $V_1(N^0)=\{N:|N|\leq 1,\ N_1\in[-1, -1+\kappa]$ for some $\kappa>0\}$,*) and a constant D, such that for any $N\in\lambda V_1(N^0)$ with any real λ, any $x\in U_1(0)$, and any real vector $(\xi,\tau)\in\Xi^n\times R^1$, $\tau\geq 1$

$$(4.1)\quad K^2(\xi+i\tau N)\leq D\{|P_0(x, \xi+i\tau N)|^2+\tau^2|N|^2|P_0^{(1)}(x, \xi+i\tau N)|^2\}$$

holds, where $K^2(\xi)$ is $\sum_{j=1}^{n}|\xi_j|^{2m_j}$.

Proof. Let be a compact set $S=\{(\xi,\tau);\ K^2(\xi+i\tau N^0)=1\}$ in $\Xi^n\times R^n$. On S and at $x=0$ if τ is zero, the 1st term of the right of (4.1) does not vanish by II. If τ is not zero and the first term is zero, by III the 2nd term does not vanish for $\tilde{\xi}\neq 0$ real, and by I (3) the 1st term does not vanish for $\tilde{\xi}=0$. Therefore on S, the right of (4.1) is positive. For any (ξ,τ), $\tau\geqq 1$ for which the value of $K^2(\xi+i\tau N^0)$ is t^2, by setting $\xi_j=\eta_j t^{\frac{1}{m_j}}$, $\tau=\sigma t^{\frac{1}{m_1}}$, (η,σ) is on S. This is from the fact that $K^2(\xi+i\tau N^0)$ equals to $K^2(\eta+i\sigma N^0)t^2$. We call this property of $K^2(\xi)$ *m-homogeneity*. We obtain thus for a constant C

$$(4.2)\quad K^2(\xi+i\tau N^0)\leq C\{|P_0(0, \xi+i\tau N^0)|^2+\tau^2|P_0^{(1)}(0, \xi+i\tau N^0)|^2\}.$$

Furthermore by I (2) and m-homogeneity of $K^2(\xi)$, we can easily get for some neighborhood $U(0)$ of $x=0$ and other constant C

$$(4.3)\qquad K^2(\xi+i\tau N^0)\leq C\{|P_0(x, \xi+i\tau N^0)|^2+\tau^2|P_0^{(1)}(x, \xi+i\tau N^0)|^2\}\ \text{for}\ x\in U(0).$$

On S by continuity with respect to N of $P_0(x, \xi+i\tau N)$ and $P_0^{(1)}(x, \xi+i\tau N)$ for any $\varepsilon>0$ there exists a neighborhood $V'(N^0)$ which is a cone

*) By $|N|^2$ we denote $\sum_{j=1}^{n}N_j^2$.

containing N^0 in its interior such that for $\nu=0,1$

(4.4) $\quad \operatorname{Sup}\{|P_0^{(\nu)}(x, \xi+i\tau N^0)-P_0^{(\nu)}(x, \xi+i\tau N)| : x\in\Omega \ (\xi, \tau)\in S\}<\varepsilon.$

Then replacing $P_0^{(\nu)}(x, +i\tau N^0)$ in (4.3) for $P_0^{(\nu)}(x, \xi+i\tau N)+\{P_0^{(\nu)}(x, \xi+i\tau N^0)-P_0^{(\nu)}(x, \xi+i\tau N)\}$, we obtain for new constant C and any (ξ, τ) on S,

(4.5) $$0<C<|P_0(x, \xi+i\tau N)|^2+\tau^2|P_0^{(1)}(x, \xi+i\tau N)|^2.$$

To prove for any (ξ, τ) $\tau\geqq 1$, we first remark that $V'(N^0)$ contains a neighborhood $V''(N^0)$ which is for some $\kappa>0$ the set $\{N: |N|\leq 1$ and $N_1\in[-1, -1+\kappa]\}$. Taking $t^2=K^2(\xi+i\tau N^0)$ and setting $\xi_j=t^{\frac{1}{m_j}}\eta_j$, $\tau=t^{\frac{1}{m_1}}\sigma$, we obtain for any $N\in V''(N^0)$

$$(\xi+i\tau N)=(t^{\frac{1}{m_1}}(\eta_1+i\sigma N_1),\ t^{\frac{1}{m_2}}(\eta_2+i\sigma l_2N_2),\cdots,\ t^{\frac{1}{m_n}}(\eta_n+i\sigma l_nN_n))$$

where l_j denotes $t^{\frac{1}{m_1}-\frac{1}{m_j}}$. Thus by I (1) and $t\geq 1$ (for $\tau\geq 1$), we obtain $0<l_j\leq l$. Hence $N'=(l_1N_1, l_2N_2,\cdots, l_nN_n)$ is in $V''(N^0)$. Applying (4.5), for $\xi=\eta$, $\tau=\sigma$, $N=N'$ and using m-homogeneity of the both hand sides we get for a constant C

(4.6) $$K^2(\xi+i\tau N^0)\leq C|P_0(x, \xi+i\tau N)|^2 +\tau^2|P_0^{(1)}(x, \xi+i\tau N)|^2\} \text{ for any } (\xi, \tau),\ \tau\geqq 1.$$

Furthermore it is clear by similar argument as above that there exists a neighborhood $V'''(N^0)$ with the same type as $V''(N^0)$ such that $\frac{1}{2}K^2(\xi+i\tau N)\leq K^2(\xi+i\tau N^0)$ for any $N\in V'''(N^0)$. Setting $U_1(0)=U(0)$ and $V_1(N^0)=V''(N^0)\cap V'''(N^0)$, we get (4.1). The proof is complete.

Next we shall state results of Trèves and their modifications for our form of operators.

Lemma 2. Let be $T(u, v)=\int u\bar{v}\exp\left(\sum_{j=1}^n t_j^2x_j^2\right)dx$ for u and v in $C_0^\infty(\Omega)$.

(1) $P(D)$ is operator with constants coefficients of order m_1. Then $T(P(D)u, P(D)u)=\sum_{\alpha\geq 0}\frac{2^{|\alpha|}}{\alpha!}t^{2\alpha}\ T(\bar{P}^{(\alpha)}(\delta), \bar{P}^{(\alpha)}(\delta))$ holds, where $t^{2\alpha}$ denotes $t_2^{2\alpha_1}, t_2^{2\alpha_2}\cdots t_n^{2\alpha_2}$, and δ does an adjoint of D; $\delta_j=D_j-2it_j^2$, and $\bar{P}(\xi)$ does $\bar{P}(\xi)=\sum_\alpha \bar{a}_\alpha\xi^\alpha$; $\bar{a}$ is the complex conjugate of a, and $\alpha!$ denotes $\alpha_1!\ \alpha_2!\cdots\alpha_n!$ Furthermore the expression of the above formula is unique.

(2) $t^{2\alpha}T(P^{(\alpha)}(D)u, P^{(\alpha)}(D)u)\leqq 2^{m_1-|\alpha|}\alpha!\ T(P(D)u, P(D)u)$ holds.

(3) $T(P_0(x, D)u, P_0(x, D)u)=\sum_{\alpha\geq 0}\frac{2^{|\alpha|}}{\alpha!}t^{2\alpha}\ T(\bar{P}_0^{(\alpha)}(x, \delta)u,\ \bar{P}_0^{(\alpha)}(x, \delta)u)+R$, where for all $t_j\geq 1$ $|R|^2\leq CT_1(u, u)\sum_{j=1}^n T_{1-\frac{1}{m_j}}(u, u)$; $T_s(u, u)=\sum_{|\alpha:m|=s}T(D^\alpha u, D^\alpha u)$ holds.

(4) $t^{2\alpha}T(P_0^{(\alpha)}(x, D)u,\ P_0^{(\alpha)}(x, D)u)\leq C\Big[T(P_0(x, D)u,\ P_0(x, D)u)$

$+\{T_1(u,u)\}^{\frac{1}{2}}\times\left\{\sum_{j=1}^{n}T_{1-\frac{1}{m_j}}(u,u)\right\}^{\frac{1}{2}}\Bigg]$ holds, for $u\in C_0^\infty(\Omega)$ and constants C's independent of u.

An outline of proof. (1) and (2) are due to Trèves. (See ([6].) For (3) and (4) we only remark some points to modify the proof of Hörmander for an operator with a homogeneous order. (See Th. I of [1].) In a term of the left of (3); $T(a_\alpha(x)D^\alpha u, b_\beta(x)D^\beta u)$, we transfer D^α from left to right and D^β from right to left by integrating by parts: $T(u, D_j u)=T(\delta_j u, u)$, for $u\in C_0^\infty(\Omega)$, and by using an almost commutative relation between D_j and δ_k: $\delta_j D_j - D_j\delta_j = 2t_j^2$, $\delta_j D_k - D_k\delta_j = 0$ for $j\neq k$. In doing so, the sum of the orders of the derivation D and δ and of the derivation of $\overline{b_\beta(x)}\times a_\alpha(x)$ and of t, which are contained in one term, is invariant. But in our case the length of α and β are not equal, therefore the regularity of a_α and b_β must be raised to $C^{2|m|}(\Omega)$, though in [1] it was sufficient for them in $C^1(\Omega)$. In these process of the integration by parts we classify the terms in each of which $\overline{b}_\beta a_\alpha$ is derivated once at least and other terms, and the former is denoted by R. Then we get

$$T(P_0(x,D)u, P_0(x,D)u)=\sum_{\alpha,\beta}C_{\alpha\beta}(a,b,t)T(\delta^\alpha u, \delta^\beta u)+R, \tag{4.7}$$

where R is the sum of terms $t^{2\gamma}T(D^\rho(\overline{b}_\beta a_\alpha)D^{\alpha'}u, D^{\beta'}u)$; $|\gamma+\alpha':m|<1$, $|\gamma+\beta':m|\leq 1$, $\rho>0$,*) and $C_{\alpha\beta}(a,b,t)$ is a quadratic form of $a_\alpha(x)$ with polynomial coefficients of t. And by the uniqueness of the representation of (1) the 1st term of (4.7) becomes that of (3). It only remains to estimate R. For $\rho>0$ fixed, for any β such that $|\beta:m|=s-|\rho:m|$ is satisfied, it is easily verified that there exist at least one α, satisfying $|\alpha:m|=s$, and a constant $C(\alpha,\rho)$ such that

$$T(D^\beta u, D^\beta u)\leq C(\alpha,\rho)t^{-2\rho}T(D^\alpha u, D^\alpha u) \tag{4.8}$$

holds for $u\in C_0^\infty(\Omega)$, by virtue of (2) of this lemma. Hence for a new constant $C'(\alpha,\rho)$

$$T(D^\beta u, D^\beta u)\leq C'(s,\rho)t^{-2\rho}T_s(u,u) \tag{4.9}$$

holds for $u\in C_0^\infty(\Omega)$. For multi-integers α' and γ such that $|\alpha'+\gamma:m|<1$, there exists at least one j such that $|\alpha'+\gamma:m|\leqq 1-\dfrac{1}{m_j}$ holds, hence there exists at least a multi-integer ρ such that $\rho\geq\gamma$ and $|\rho+\alpha':m| =1-\dfrac{1}{m_j}$ hold. Then applying (4.9) for $\beta=\alpha'$ $s=1-\dfrac{1}{m_j}$, we get

$$T(D^{\alpha'}u, D^{\alpha'}u)\leq C(j,\rho)t^{-2\rho}T_{1-\frac{1}{m_j}}(u,u)\leq C(\rho)t^{-2\rho}\sum_{j=1}^{n}T_{1-\frac{1}{m_j}}(u,u). \tag{4.10}$$

For β' and γ such that $|\beta'+\gamma:m|\leq 1$ holds, there exists at least one σ such that $\sigma\geq\gamma$ and $|\sigma+\beta':m|=1$ hold. Then similarly we get

$$T(D^{\beta'}u, D^{\beta'}u)\leq C(\sigma)t^{-2\sigma}T_1(u,u). \tag{4.11}$$

Using (4.10), (4.11) and Schwarz inequality, we get

*) $\rho=(\rho_1,\rho_2,\cdots\rho_n)$ where ρ_j are non-negative integers, is called multi-integer, and sometimes denoted by $\rho\geqq 0$. If there is at least one positive ρ_j, it is denoted by $\rho>0$.

$$(4.12) \qquad |R|^2 \leq Ct^{2(\gamma-\rho)+(\gamma-\sigma)} T_1(u, u) \sum_{j=1}^{n} T_{1-\frac{1}{m_j}}(u, u)$$

for a constant C independent of t and $u \in C_0^\infty(\Omega)$. Thus if all t_j are $\geqq 1$, $Ct^{2(\gamma-\rho)+2(\gamma-\sigma)}$ in the right replaced by an other constant independent of t. Thus (3) is proved. To obtain (4), we apply (3) for $P_0(x, D) = P_0^{(\beta)}(x, D)$ and use (4.9), (4.10) and (4.11) and similar calculation as in [1] is allowed. For References, see the next article.