138. *On Concircular Scalar Fields*

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In a previous paper [1], the author determined complete Riemannian manifolds admitting a concircular scalar field [1, Theorem 1] and, in particular, those admitting a special concircular scalar field [1, Theorem 2]. If M is an n-dimensional Riemannian manifold with metric tensor field $g_{\mu\lambda}$, and we denote by $\{{}^{\kappa}_{\mu\lambda}\}$ the Christoffel symbol and by ∇ the covariant differentiation with respect to $\{{}^{\kappa}_{\mu\lambda}\}$, then a *concircular scalar field* ρ is by definition a scalar field satisfying the differential equation

(1) $$\nabla_\mu \nabla_\lambda \rho = \phi g_{\mu\lambda} ,$$

where ϕ is a scalar field and will be called the *characteristic function* of ρ. A *special* concircular scalar field is by definition a concircular one satisfying

(2) $$\nabla_\mu \nabla_\lambda \rho = (-k\rho + b) g_{\mu\lambda}$$

with constant coefficients k and b. We shall call k the *characteristic constant* of ρ.

In the present paper we shall show that, if a Riemannian manifold admits functionally independent concircular scalar fields, then they are special concircular scalar fields having the same characteristic constant, and that a concircular scalar field, which is not invariant under an infinitesimal isometry, is also a special concircular one. In this light, we may say that the special concircular scalar field is *not* so special.

A point is called a stationary or ordinary point of ρ according as the gradient vector field $\rho_\lambda = \partial_\lambda \rho$ vanishes there or not. We notice that the characteristic function ϕ of ρ is a differentiable function of ρ itself in a neighborhood of an ordinary point of ρ. We shall first show the following

Lemma 1. *If ρ and σ are concircular scalar fields functionally dependent of each other, then σ is linear in ρ with constant coefficients in a neighborhood of an ordinary point of ρ.*

Proof. Suppose that σ satisfies the equation

(3) $$\nabla_\mu \sigma_\lambda = \psi g_{\mu\lambda} ,$$

where $\sigma_\lambda = \partial_\lambda \sigma$. By our assumption, we may put $\sigma_\lambda = A \rho_\lambda$, A being a proportional factor. By substituting this into (3) and using (1), we have

$$(\partial_\mu A) \rho_\lambda = (\psi - A\phi) g_{\mu\lambda} .$$

Since the matrix $(g_{\mu\lambda})$ is non-singular, it follows that $\partial_\mu A=0$ and hence A is a constant. Thus σ is of the form

$$\sigma=A\rho+B\ . \qquad \text{Q.E.D.}$$

Applying Ricci's identity to (1), we obtain the equation

$$K_{\nu\mu\lambda}{}^{\kappa}\rho_\kappa=K_{\nu\mu\lambda\kappa}\rho^\kappa=-\frac{d\phi}{d\rho}(\rho_\nu g_{\mu\lambda}-\rho_\mu g_{\nu\lambda})\ , \tag{4}$$

where $K_{\nu\mu\lambda}{}^{\kappa}$ is the curvature tensor field of M. If ρ is, in particular, a special concircular scalar field, then we have

$$K_{\nu\mu\lambda\kappa}\rho^\kappa=k(\rho_\nu g_{\mu\lambda}-\rho_\mu g_{\nu\lambda})\ . \tag{5}$$

We shall have the following

Theorem 1. *If two concircular scalar fields ρ and σ are functionally independent of each other, then they are special concircular scalar fields having the same characteristic constant.*

Proof. The function ψ in (3) is a differentiable function of σ and we obtain the equation

$$K_{\nu\mu\lambda\kappa}\sigma^\kappa=-\frac{d\psi}{d\sigma}(\sigma_\nu g_{\mu\lambda}-\sigma_\mu g_{\nu\lambda}) \tag{6}$$

similar to (4). Contracting (4) with σ^λ and (6) with ρ^λ respectively and taking account of skew-symmetry of $K_{\nu\mu\lambda\kappa}$ in κ and λ, we have

$$\left(\frac{d\phi}{d\rho}-\frac{d\psi}{d\sigma}\right)(\sigma_\nu\rho_\mu-\sigma_\mu\rho_\nu)=0\ .$$

Since ρ and σ are functionally independent, we see that the first factor should vanish and $\frac{d\phi}{d\rho}=\frac{d\psi}{d\sigma}$ is equal to a constant, say $-k$. Thus ϕ and ψ are linear functions of ρ and σ respectively.

Theorem 2. *If an n-dimensional Riemannian manifold M admits $n-1$ concircular scalar fields $\rho_a(a=1,\cdots,n-1)$ and they are functionally independent, then the manifold M is of constant sectional curvature.*

Proof. By means of Theorem 1, we have now the equations

$$[K_{\nu\mu\lambda\kappa}-k(g_{\nu\kappa}g_{\mu\kappa}-g_{\mu\kappa}g_{\nu\lambda})]\rho_a^\kappa=0$$

from equations similar to (5). Since the matrix (ρ_a^κ) is of rank $n-1$, we can put

$$K_{\nu\mu\lambda\kappa}-k(g_{\nu\kappa}g_{\mu\lambda}-g_{\mu\kappa}g_{\nu\lambda})=L_{\nu\mu\lambda}H_\kappa\ ,$$

where $L_{\nu\mu\lambda}$ and H_κ are proportional factors. However, by virtue of skew-symmetry of the left hand side in κ and λ, the above equation should be equal to zero, that is,

$$K_{\nu\mu\lambda\kappa}=k(g_{\nu\kappa}g_{\mu\lambda}-g_{\mu\kappa}g_{\nu\lambda})$$

and M is of constant sectional curvature. Q.E.D.

In a compact manifold M, there exist exactly two stationary points of a concircular scalar field [1, Theorem 1, C)]. Moreover, the only special concircular scalar field admitted in a compact manifold

M is one having a positive characteristic constant, and the manifold M is then a spherical space [1, Theorem 2, III)]. It follows from these facts that

Theorem 3. *If a compact Riemannian manifold M admits functionally independent concircular scalar fields, then M is a spherical space.*

Next let a vector field v^κ be an infinitesimal isometry and denote by $\pounds$ the Lie differentiation with respect to v^κ. Then we know the equations

$$\pounds g_{\mu\lambda} = \nabla_\mu v_\lambda + \nabla_\lambda v_\mu = 0 \tag{7}$$

and

$$\pounds\left\{{}^{\kappa}_{\mu\lambda}\right\} = \nabla_\mu \nabla_\lambda v^\kappa + v^\nu K_{\nu\mu\lambda}{}^\kappa = 0 \tag{8}$$

and that $\pounds$ commutes with the covariant differentiation ∇. We shall prove the following

Theorem 4. *If a concircular scalar field ρ is not invariant under an infinitesimal isometry v^κ, that is, $\pounds\rho \neq 0$, then ρ is a special concircular scalar field.*

Proof. Applying $\pounds$ to (1), we have

$$\nabla_\mu \nabla_\lambda \pounds\rho = (\pounds\phi) g_{\mu\lambda} \tag{9}$$

and see that $\pounds\rho$ is also a concircular scalar field. If $\pounds\rho$ is functionally independent of ρ, then the theorem follows from Theorem 1. If $\pounds\rho$ is functionally dependent of ρ, then we can put

$$\pounds\rho = v^\kappa \rho_\kappa = A\rho + B \tag{10}$$

by means of Lemma 1. Differentiating covariantly (10) and taking account of (1), we have

$$(\nabla_\lambda v^\kappa)\rho_\kappa + \phi v_\lambda = A\rho_\lambda\ , \tag{11}$$

and, further differentiating (11) and taking account of (7), (8), and (1),

$$-v^\nu K_{\nu\mu\lambda\kappa}\rho^\kappa + (\partial_\mu \phi) v_\lambda = A\phi g_{\mu\lambda}\ .$$

By contraction of this equation with ρ^λ and substitution of (10), we obtain

$$(\partial_\mu \phi)(A\rho + B) = A\phi\rho_\mu\ .$$

The solution of this equation is given by the form

$$\phi = -k\rho + b$$

with constant coefficients k and b. Thus ρ is a special concircular scalar field. Q.E.D.

By the same reason as that for Theorem 3, we have

Theorem 5. *If a compact Riemannian manifold M admits an infinitesimal isometry v^κ and a concircular scalar field which is not invariant under v^κ, then the manifold M is a spherical space.*

Reference

[1] Y. Tashiro: Complete Riemannian manifolds and some vector fields. Trans. Amer. Math. Soc., **117**, 251-275 (1965).