214. Fubini Theorems for Generalized Lebesgue-Bochner-Stieltjes Integral

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Let R be the space of reals. If Y_i, W $(i=1, \cdots, k)$ are seminormed spaces then by $L(Y_1, \cdots, Y_k; W)$ we shall denote the space of all operators u which are k-linear and continuous from the product of the spaces Y_i $(i=1, \cdots, k)$ into the space W. The seminorm of elements in the above spaces will be denoted by $| \ |$.

A family of sets V of an abstract space X will be called a pre-ring if for any two sets $A_1, A_2 \in V$ we have $A_1 \cap A_2 \in V$, and there exists disjoint sets $B_1, \cdots, B_k \in V$ such that $A_1 \backslash A_2 = B_1 \cup \cdots \cup B_k$.

A nonnegative function v on the pre-ring V will be called a volume if for every countable family of disjoint sets $A_t \in V$ $(t \in T)$ such that $A = \sum_T A_t \in V$ we have $v(A) = \sum_T v(A_t)$.

A triple (X, V, v) where V is a pre-ring of sets of X and v is a volume on V, will be called a volume space. If the triples (X_i, V_i, v_i) $(i=1, \cdots, k)$ are volume spaces then the triple (X, V, v) defined by $X = X_1 \times \cdots \times X_k$ and $V = V_1 \times \cdots \times V_k$ consisting of all sets of the form $A = A_1 \times \cdots \times A_k$; $A_i \in V_i$ with $v(A) = v_1(A_1) \cdots v_k(A_k)$ is a volume space.

Let (X, V, v) be a fixed volume space. Denote by $M_q(v, Z)$ $(1 < q \leq \infty)$ the space of all finite additive functions μ from the pre-ring V into a Banach space Z and such that $\mu(A)=0$ if $v(A)=0$ and

$$\sup \{(\textstyle\sum_n |\mu(A_n)|^q v(A_n)^{1-q})^{1/q}\} = \|\mu\|_q < \infty$$

when $q \neq \infty$, where the supremum is taken over all finite families of disjoint sets $A_n \in V$ such that $v(A_n) > 0$. In the case when $q = \infty$ let $\sup \{|\mu(A)| v(A)^{-1} : A \in V\} = \|\mu\|_q < \infty$ where the supremum is taken over all sets $A \in V$ such that $v(A) > 0$.

Now if $1/p_i + 1/q_i = 1$, $p_i \geq 1$, $i=1,2$ and $u \in L(Y_1, Y_2, Z; W)$, denote by $M(q_i, v_i, Z, u)$ the family of all functions $\mu(A_1, A_2)$ from $V_1 \times V_2$ into Z which are additive in each variable A_i separately and $\mu(A_1, A_2)=0$ if $v_1(A_1)=0$ or $v_2(A_2)=0$; moreover assume that the following norm is finite $\|\mu\| = \sup \{|\sum_{ij} u(y_{1i}, y_{2j}, \mu(A_{1i}, A_{2j}))(v_1(A_{1i}))^{-1/p_1}(v_2(A_{2j}))^{-1/p_2} a_{1i} a_{2j}|\}$ where the supremum is taken over all finite systems such that $\|y_{1i}\| \leq 1$ $\|y_{2j}\| \leq 1$, $\sum |a_{1i}|^{p_1} \leq 1$, $\sum |a_{2j}|^{p_2} \leq 1$, where A_{1i} is a family of disjoint sets of the pre-ring V_1 such that $v_1(A_{1i}) > 0$ and similarly A_{2j} is a finite family of disjoint sets of the pre-ring V_2 such that $v_2(A_{2j}) > 0$.

If $q=q_1=q_2$ and $u(y_1, y_2, z)=z(y_1, y_2)$ for $y_i \in Y_i$, $z \in L(Y_1, Y_2; W)$ then we have $M_q(v, Z) \subset M(q, q, v_1, v_2, Z, u)$.

Theorem 1. Let (X, V, v) be the product volume space of the volume spaces (X_i, V_i, v_i) $(i=1, \cdots, k)$. If $\mu_i \in M_q(v_i, Z_i)$ where $1< q\leq \infty$ and $u \in L(Z_i, \cdots, Z_k; W)$ then $\mu \in M_q(v, W)$ where

$$\mu(A_1\times \cdots \times A_k)=u(\mu_1(A_1), \cdots, \mu_k(A_k)) \qquad \text{for } A \in V$$

Let (X, V, v) be a volume space and Y be a fixed Banach space. Denote by $S(V, Y)=S(Y)$ the set of all functions of the form $h=y_1\chi_{A_1}+\cdots+y_k\chi_{A_k}$ where $y_i \in Y_i$ and $A_i \in V$ are disjoint sets. Put $\|h\|=|y_1|\, v(A_1)+\cdots+|y_k|\, v(A_k)$.

A sequence of functions s_n is called basic if there exist a sequence $h_n \in S(Y)$ and a constant $M>0$ such that $s_n=h_1+\cdots+h_n$, $\|h_n\|\leq M4^{-n}$ for $n=1, 2, \cdots$

A set $A\subset X$ is called a null set if for every $\varepsilon>0$ there exists a countable family of sets $A_t \in V$ $(t \in T)$ such that $A\subset \bigcup_T A_t$ and $\sum_T v(A_t)<\varepsilon$.

A condition $c(x)$ depending on a parameter $x \in A_0\subset X$ is said to be satisfied almost everywhere on the set A_0 if there exists a null set A such that condition is satisfied at every point of the set $A_0\backslash A$.

Denote by $L_1(v, Y)$ the space of all functions f such that there exists a basic sequence s_n convergent almost everywhere on the space X to the function f. Put $\|f\|=\lim \|s_n\|$. This definition is correct, that is, it doesn't depend on the particular choice of the basic sequence. It follows from Theorem 1 [1], that the space $(L_1(v, Y), \|\ \|)$ is a complete seminormed space. The set of simple functions $S(V, Y)$ is dense in the space $L_1(v, Y)$ according to Lemmas 1 and 4 [1].

Now let $1\leq p<\infty$. Denote by a the function $a(y)=|y|^{p-1}y$ for $y \in Y$. Since the function and its inverse $a^{-1}(y)=|y|^{1/p-1}y$ for $y \in Y$ are continuous on the space Y therefore it establishes a homeomorphism of the space onto itself.

Denote by $L_p(v, Y)$ the space of all functions f from the set X into the space Y such that $a\circ f \in L_1(v, Y)$. Put

$$\|f\|_p=\left(\int |a\circ f|\, dv\right)^{1/p}=\left(\int |f(x)|^p\, dv\right)^{1/p}.$$

The space $(L_p(v, Y), \|\ \|_p)$ is a complete seminormed space and the set $S(V, Y)$ is dense in it according to Theorem 1 [4].

Now let (X, V, v) be the product space of the volume spaces (X_i, V_i, v_i) $(i=1, 2)$. Take any simple functions $s_i \in S(V_i, Y_i)$ and assume that $s_i=\sum_{n_i} y_{n_i}\chi_{A_{n_i}}$. Let $\mu \in M_q(v, Z)$ and let u be a multilinear continuous operator from the product of the Banach spaces Y_1, Y_2, Z into a Banach space W. Define

$$\int u(s_1, s_2, d\mu) = \sum_{n_1, n_2} u(y_{n_1}, y_{n_2}, \mu(A_{n_1} \times A_{n_2})).$$

It is easy to see that the definition is correct. Put $U = L(Y_1, Y_2, Z; W)$. The integral operator just defined is linear in each variable u, s_1, s_2, μ separately and is defined on a dense set of the product of the spaces U, $L_p(v_1, Y_1)$, $L_p(v_2, Y_2)$, $M_q(v, A)$, where $1 < p \leq \infty$ and $1/p + 1/q = 1$. Now from the inequality

$$\left|\int u(s_1, s_2, d\mu)\right| \leq |u| \, \|s_1\|_p \, \|s_2\|_p \, \|\mu\|_q$$

and from the completeness of the space W we get that there exists a unique extension of the operator to a multilinear continuous operator defined on $U \times L_p(v_1, Y_1) \times L_p(v_2, Y_2) \times M_q(v, Z)$.

In a similar way one could define the integral operator $\int u_0(f, d\mu)$ for $f \in L_p(v, Y)$, $\mu \in M_q(v, X)$, $u_0 \in L(Y, Z; W)$. When it is important to indicate the variable of integration which shall use the symbol $\int u_0(f(x), \mu(dx))$.

Fubini's Theorem for the integral $\int u(f_1, f_2, d\mu)$

Take any multilinear continuous operator $u \in L(Y_1, Y_2, Z; W) = U$. Define an operator $u_1(y_2, z) = u(\cdot, y_2, z)$ for $y_2 \in Y_2, z \in Z$. We see that $u_1 \in L(Y_2, Z; Z_0) = U_1$ where $Z_0 = L(Y_1, W)$. Define also the operator $u_0(y_1, z_0) = z_0(y_1)$ for $y_1 \in Y_1$, $z_0 \in Z_0$. We have $u_0 \in L(Y_1, Z_0; W)$ and $|u| = |u_1|$, $|u_0| = 1$.

Let (X, V, v) be the product volume space of the volume spaces (X_i, V_i, v_i) $(i=1, 2)$. Assume that $1 \leq p < \infty$ and $1/p + 1/q = 1$. We have the following theorem.

Theorem 2. (1) If $\mu \in M_q(v, Z)$ then for all $A_1 \in V_1$ the vector function μ_{A_1} defined by the formula

$$\mu_{A_1}(A_2) = \mu(A_1 \times A_2) \text{ for all } A_2 \in V_2$$

belongs to the space $M_q(v_2, Z)$.

(2) The operator $\mu_1 = r(f_2, \mu)$ defined by means of the integral

$$\mu_1(A_1) = \int u_1(f_2, d\mu_{A_1}) \quad \text{for all } A_1 \in V_1$$

is bilinear from the product $L_p(v_2, Y_2) \times M_q(v, Z)$ into the space $M_q(v, Z_0)$ and

$$\|\mu_1\|_q \leq |u| \, \|f_2\|_p \, \|\mu\|_q \qquad \text{for all } f_2 \in L_p(v_2, Y_2), \mu \in M_q(v, Z).$$

(3) Moreover the following equality holds

$$\int u(f_1, f_2, d\mu) = \int u_0(f_1, dr(f_2, \mu))$$

for all $f_i \in L_p(v_i, Y_i)$ $(i=1, 2)$, $\mu \in M_q(v, Z)$.

(The above theorem can be easily generalized to the case when $f_1 \in L_{p_1}(v_1, Y_1)$, $f_2 \in L_{p_2}(v_2, Y_2)$, and $\mu \in M(q_1, q_2, v_1, v_2, Z, u) = M$.

If we take the trilinear operator $u(y_1, y_2, z)=z(y_1, y_2)$ for $y_i \in Y_i$, $z \in Z$ and define $Z=L(Y_1, Y_2; W)$, then the space M is isomorphic and isometric to the space of all bilinear continuous operators h from the product $L_{p_1}(v_1, Y_1)\times L_{p_2}(v_2, Y_2)$ into the space W).

Consider the following example. Let Y_i, Z, W be equal to the space C of complex numbers. Let $u(y_1, y_2, z)=y_1y_2z$. Then we have $u_1(y_2, z)=y_2z$ and $u_0(y_1, z_0)=y_1z_0$. If $f_i \in L_p(v_i, C)$, $\mu \in M_q(v, C)$ then we get from the theorem

$$\int f_1(x_1)f_2(x_2)\mu(dx_1\times dx_2)=\int f_1(x_1)\mu_1(dx_1)$$

where $\mu_1(A_1)=\int f_2(x_2)\mu(A_1\times dx_2)$ for all $A_1 \in V_1$.

Fubini's theorem for generalized Lebesgue-Bochner-Stieltjes integral.

Denote by (X, V, v) the product volume space of the volume spaces (X_i, V_i, v_i). Let $1\leq p<\infty$ and $1/p+1/q=1$.

Let Y, Z_1, Z_2, W be Banach spaces. Assume that $u \in U=L(Y, Z_1, Z_2; W)$ and define a new operator $u_1(y, z_2)=u(y, \cdot, z_2)$ for $y \in Y$, and $z_2 \in Z_2$. We see that $u_1 \in L(Y, Z_2; Y_1)$, where $Y_1=L(Z_1; W)$. Define $u_0(y_1, z_1)=y_1(z_1)$ for $y_1 \in Y_1$ and $z_1 \in Z_1$. Notice that $u_0 \in L(Y_1, Z_1; W)$ and $|u|=|u_1|$ and $|u_0|=1$.

Put $N=\{f \in L_p(v_1, Y_1): ||f||_p=0\}$. The set N is linear and according Theorem 1 [1], coincides with the set of all functions f from the set X_1 into the space Y_1 such that $f(x)=0$ v_1-a.e.

Consider the quotient space $L_p(v_1, Y_1)/N$ and define the norm of a class $[f]=f+N$ by $||[f]||_p=||f||_p$. This definition is correct. Notice that in order to determine a class $[f]$ it is enough to give the values of the function $f(x_1)$ v_1-almost everywhere.

Since the integral operator $\int u_0(f, d\mu)$ is linear in the variable f, and we have the estimation

$$\left|\int u_0(f, d\mu)\right|\leq|u_0|\ ||f||_p\ ||\mu||_q,$$

therefore the following definition

$$\int u_0([f], d\mu)=\int u_0(f, d\mu)$$

is correct where $[f] \in L_p(v_1, Y_1)/N$. The operator defined in this way $\int u_0(g, d\mu)$ is bilinear and we have

$$\left|\int u_0(g, d\mu)\right|\leq|u_0|\ ||g||_p\ ||\mu||_q$$

where $g \in L_p(v_1, Y_1)/N$ and $\mu \in M_q(v_1, Z_1)$.

Theorem 3. (1) If $f \in L_p(v, Y)$, there exists a v_1-null set C such that $f(x_1, \cdot) \in L_p(v_2, Y)$ if $x_1 \notin C$.

(2) The operator $\bar{f}_1=r(f, \mu_2)$ defined by the formula

$$\bar{f}(x_1)=\int u_1(f(x_1,\ \cdot),\ d\mu_2 \qquad \text{if } x_1 \notin C$$

is bilinear from the product $L_p(v,\ Y)\times M_q(v_2,\ Z_2)$ into the space $L_p(v_1,\ Y_1)/N$ and

$$\|\bar{f}\|_p \leq |u|\ \|f\|_p\ \|\mu_2\|_q$$

for all $f\in L_p(v,\ Y)$ and $\mu_2\in M_q(v_2,\ Z_2)$.

(3) Moreover $\int u(f,\ d\mu_1,\ d\mu_2)=\int u_0(r(f,\ \mu_2),\ d\mu_1)$ for all $f\in L_p(v,Y)$, $\mu_i\in M_q(v_i,\ Z_i)$ $(i=1,\ 2)$.

Consider the following example. Let $Y=Z$ be a complex Banach space and let $Z_1=Z_2=C$ be the space of complex numbers. Define $u(y,\ z_1,\ z_2)=z_1z_2y$ for all $z_i\in C,\ y\in Y$. We see that we may identify $Y_1=W$. Thus we have $u_1(y,\ z_1)=yz_1$ and also $u_0(y,\ z_1)=z_1y$.

Now if $f\in L_p(v,\ Y)$ and $\mu_i\in M_q(v_i,\ C)$ then $f(x_1,\ \cdot)\in L_p(v_2,\ Y)$ for v_1-almost all $x_1\in X_1$. For the function $h(x_1)=\int f(x_1,\ \cdot)d\mu_2$ we have $h\in L_p(v_1,\ Y)$ and

$$\int hd\mu_1=\int\left(\int f(x_1,\ x_2)\mu_2(dx_2)\right)\mu_1(dx_1)=\int fd(\mu_1\times\mu_2).$$

For the case $p=1$ we get the classical Fubini theorem for Bochner summable functions (compare Dunford and Schwartz: Linear Operators, p. 193).

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