217. *A Class of Markov Processes with Interactions.* II

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Here, we look at the branches which describe the interactions between particles of the model in [4]. This leads to finer proofs of Chapman-Kolmogorov equation and the backward equation. A consistency condition holds for probabilities of events which are determined by bundles of these branches.

1. To consider the simplest model with binary interactions, let $q(t,y)\equiv q_1(t,y)$ and $q_0\equiv q_2\equiv q_3\equiv\cdots\equiv 0$, and write $\pi(y'\,|\,t,y,E)$ for $\pi_1(y_1\,|\,t,y,E)$ in **1** of [4].[1] Then, the forward and the backward equations are

$$P^{(f)}(s,x,t,E)=P_0(s,x,t,E)+\int_s^t d\tau\int_{R^2}P^{(f)}(s,x,\tau,dy)\times P^{(f)}_{s,\tau}(dy')q(\tau,y)\int_R\pi(y'\,|\,\tau,y,dz)P_0(\tau,z,t,E),\tag{1}$$

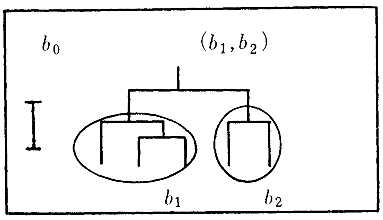
$$P^{(P^{(f)}_{s_0s})}(s,x,t,E)=P_0(s,x,t,E)+\int_s^t d\tau\int_{R^2}P_0(s,x,\tau,dy)\times P^{(f)}_{s_0\tau}(dy')q(\tau,y)\int_R\pi(y'\,|\,\tau,y,dz)P^{(P^{(f)}_{s_0\tau})}(\tau,z,t,E),\tag{2}$$

where $P^{(f)}_{s,\tau}(E)\doteq\int_R f(dx)P^{(f)}(s,x,\tau,E),\quad s_0\le s\le t.$

Let T be the set of all branches which grow downward with binary branching points and the trivial branch (or a pole) b_0. For b_1 and b_2 in T, $b=(b_1,b_2)$ is the branch which has b_1 and b_2 on the left and the right side of the highest branching point. Length $l(b)$ and the number of the end points $\sharp(b)$ are defined by

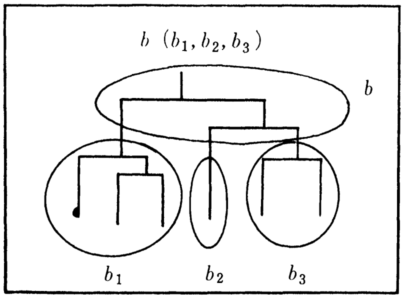
$$l(b_0)=0,\ l((b_1,b_2))=1+\max(l(b_1),\ l(b_2)),$$

$$\sharp(b_0)=1,\ \sharp((b_1,b_2))=\sharp(b_1)+\sharp(b_2).$$



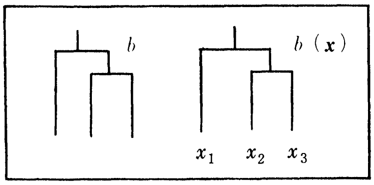
When $\sharp(b)=n$, let $b(b_1,\cdots,b_n)$ be the branch b with branches $b_1,\cdots,b_n$ connected at the end points, with b_k at the k-th end point from the left. We write $b\ge b'$ when $b=b'(b_1,\cdots,b_n)$. Since the branches $b_1,\cdots,b_n$ are determined

1) This is for the simplicity of descriptions. Results in this paper can be extended to the models in [4].



uniquely for given b and $b'(\leq b)$, we denote the bundle of branches $(b_1, \cdots \cdots, b_n)$ by b/b'. When $\sharp(b)=n$ and $\boldsymbol{x}=(x_1, \cdots x_n)$, let $b(\boldsymbol{x})=b(x_1, \cdots, x_n)$ be the branch b with variables $x_1, \cdots, x_n$ at the end points, with x_k at the k-th end point from the left. $(b_1(\boldsymbol{x}_1), b_2(\boldsymbol{x}_2))$ and $b(b_1(\boldsymbol{x}_1), \cdots, b_n(\boldsymbol{x}_n))$ are defined similarly.

For $b \in T$, $\boldsymbol{x}=(x_1, \cdots, x_{\sharp(b)})$, $\boldsymbol{s}=(s_1, \cdots, s_{\sharp(b)})$ and t such that $\max(\boldsymbol{s}) \leq t$, we define $P(\boldsymbol{s}, b(\boldsymbol{x}), t, E)$ inductively by



$$P(s_1, b_0(x_1), t, E)=P_0(s_1, x_1, t, E),$$

$$P((\boldsymbol{s}_1, \boldsymbol{s}_2), (b_1(\boldsymbol{x}_1), b_2(\boldsymbol{x}_2)), t, E)$$

$$=\int_{\max(\boldsymbol{s}_1, \boldsymbol{s}_2)}^{t} d\tau \int_{R^2} P(\boldsymbol{s}_1, b_1(\boldsymbol{x}_1), \tau, dy) \times P(\boldsymbol{s}_2, b_2(\boldsymbol{x}_2), \tau, dy') q(\tau, y) \times \int_R \pi(y' \mid \tau, y, dz) P_0(\tau, z, t, E), \tag{3}$$

where $\boldsymbol{s}_1=(s_1, \cdots, s_m)$, $\boldsymbol{s}_2=(s_{m+1}, \cdots \cdots, s_{m+n})$, $\boldsymbol{x}_1=(x_1, \cdots, x_m)$, $\boldsymbol{x}_2=(x_{m+1}, \cdots, x_{m+n})$, $m=\sharp(b_1)$, and $n=\sharp(b_2)$.[2)]

Then, by a simple induction, we have

$$P(\boldsymbol{s}, b(\boldsymbol{x}), t, R)+\int_{\max(\boldsymbol{s})}^{t} d\tau \int_R P(\boldsymbol{s}, b(\boldsymbol{x}), \tau, dy) q(\tau, y) \leq 1,$$

starting with the equality in case $b=b_0$.

2. Theorem 1. *For $\boldsymbol{s}, t, u$ such that* $\max(\boldsymbol{s}) \leq t \leq u$,

$$P(\boldsymbol{s}, b(\boldsymbol{x}), u, E)=\sum_{b' \leq b} \int_{R^{\sharp(b')}} \prod_{b_k \in b/b'} P(\boldsymbol{s}_k, b_k(\boldsymbol{x}_k), t, dy_k) \times P((t, \cdots, t), b'(\boldsymbol{y}), u, E). \tag{4}$$

Note. This is an exact extension of (52) *in Feller* [1] *to our present model*:

$$P_n(s, x, u, E)=\sum_{k=0}^{n} \int_R P_k(s, x, t, dy) P_{n-k}(t, y, u, E).$$

Outline of the proof. For $b=b_0$, (4) is the Chapman-Kolmogorov equation for $P_0(s, x, t, E)$. If we assume the result for b_1 and b_2, then for $b=(b_1, b_2)$,

$$P(\boldsymbol{s}, b(\boldsymbol{x}), u, E)=\left(\int_{\max(\boldsymbol{s})}^{t} d\tau+\int_t^u d\tau\right) \int_{R^2} P(\boldsymbol{s}_1, b_1(\boldsymbol{x}_1), \tau, dy) \times P(\boldsymbol{s}_2, b_2(\boldsymbol{x}), \tau, dy') q(\tau, y) \int_R \pi(y' \mid \tau, y, dz) P_0(\tau, z, u, E)$$

2) Intuitively, $P((s_1, \cdots, s_n), b(x_1, \cdots, x_n), t, E)$ is the probability that the particle, started at x_1 at time s_1, is in the set E at time t after the interactions with other particles which started at $x_2, \cdots, x_n$, at times $t_2 \cdots, t_n$, respectively, where the order of the interactions are determined by the branch b.

$$=\int_R P(s,b(\boldsymbol{x}),t,dy)P_0(t,y,u,E)+\int_t^u d\tau\int_{R^2}$$

$$\times\Big\{\sum_{b'\leq b_1}\int_{R^{\#(b')}}\prod_{b'_k\in b_1/b'}P(s_k,b'_k(\boldsymbol{x}_k),t,dy_k)P((t,\cdots,t),b'(\boldsymbol{y}),\tau,dy)\Big\}$$

$$\times\Big\{\sum_{b''\leq b_2}\int_{R^{\#(b'')}}\prod_{b''_j\in b_2/b''}P(s_j,b''_j(\boldsymbol{x}'_j),t,dy'_j)P((t,\cdots,t),b''(\boldsymbol{y}')\tau,dy')\Big\}$$

$$\times q(\tau,y)\int_R\pi(y'\,|\,\tau,y,dz)P_0(\tau,z,u,E)$$

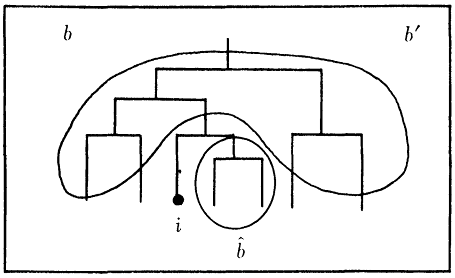
$$=\int_{R^{\#(b_0)}}P(s,b(\boldsymbol{x}),t,dy)P(t,b_0(y),u,E)+\sum_{b'\leq b_1}\sum_{b''\leq b_2}\int_{R^{\#((b',b''))}}$$

$$\times\prod_{(b'_k\in b_1/b',b''_j\in b_2/b'')}P(s_k,b'_k(\boldsymbol{x}_k),t,dy_k)P(s_j,b''_j(\boldsymbol{x}'_j),t,dy'_j)$$

$$\times P((t,\cdots,t),(b'(\boldsymbol{y}),b''(\boldsymbol{y}')),u,E).$$

But, this is the right side of (4), since $b/b_0=\{b\}$ and there are natural one to one correspondences between $\{b'\leq b_1\}\times\{b''\leq b_2\}$ and $\{b'\leq b\}-\{b_0\}$, between $\{b_1/b'\}\times\{b_2/b''\}$ and $b/(b',b'')-\{b_0\}$ for each fixed $b'\leq b_1$ and $b''\leq b_2$.[3)]

For a branch $b(\neq b_0)$ and the i-th end point of b from the left, there is a unique pair of branches b' and $\hat{b}$ such that $b(\boldsymbol{x})=b'(x_1,\cdots,x_{i-1},(b_0(x_i),\hat{b}(\hat{\boldsymbol{x}})),x_k,\cdots,x_n)$ or $b(\boldsymbol{x})=b'(x_1,\cdots,x_k,(\hat{b}(\hat{\boldsymbol{x}}),b_0(x_i)),x_{i+1},\cdots,x_n)$ for some k. This $\hat{b}$ is called *the closest subbranch of b to the i-th end point.*



Theorem 2. *By substituting $s=(s_1,s_2)$ and $b(\boldsymbol{x})=(b_1(\boldsymbol{x}_1),b_2(\boldsymbol{x}_2))$ in the place of r_1 and y_1 of $P((r_1,\cdots,r_n),\tilde{b}(y_1,\cdots y_n),t,E)$, we have*

$$P((s,r_2,\cdots,r_n),\tilde{b}(b(\boldsymbol{x}),y_2,\cdots,y_n),t,E)=\int_{\max(s,\hat{r})}^t d\tau$$

$$(5)\qquad\times\int_{R^2}P(s_1,b_1(\boldsymbol{x}_1),\tau,dy)\,P(s_2,b_2(\boldsymbol{x}_2),\tau,dy')q(\tau,y)$$

$$\times\int_R\pi(y'\,|\,\tau,y,dz)P((\tau,r_2,\cdots,r_n),\tilde{b}(z,y_2,\cdots,y_n),t,E),$$

where $\hat{r}=(r_2,\cdots,r_k)$ are time parameters which correspond to the closest subbranch $\hat{b}$ of $\tilde{b}$ to the first end point, $s_1=(s_1,\cdots,s_{\#(b_1)})$ and $s_2=(s_{\#(b_1)+1},\cdots,s_{\#(b_1)+\#(b_2)})$.[4)]

Outline of the proof. When $\tilde{b}=b_0$, (5) coincides with (3). When $\tilde{b}=(b',b'')$, assume (5) with $\tilde{b}$ replaced by b'. Since $\hat{b}$ is also the closest subbranch of b' to the first end point, $P((s,r_2,\cdots,r_n),\tilde{b}(b(\boldsymbol{x}),y_2,\cdots,y_n),t,E)$ is equal to

3) These correspondences are of the form $b'\times b''\rightarrow(b',b'')$.

4) The substitution can take place at any end point of $\tilde{b}$, where the corresponding formulation is clear.

$$
\begin{aligned}
&P((s, r', r''), (b'(b(x), y), b''(y'')), t, E) \\
&= \int_{\max(s,r',r'')}^{t} d\sigma \int_{R^2} P((s, r'), b'(b(x), y'), \sigma, dy) P(r'', b''(y''), \sigma, dy') \\
&\quad \times q(\sigma, y) \int_R \pi(y' \mid \sigma, y, dz) p_0(\sigma, z, t, E) \\
&= \int_{\max(s,r',r'')}^{t} d\sigma \int_{R^2} \left\{ \int_{\max(s,\hat{r})}^{\sigma} d\tau \int_{R^2} P(s_1, b_1(x_1), \tau, dv) \right. \\
&\quad \times P(s_2, b_2(x_2), \tau, dv') q(\tau, v) \int_R \pi(v' \mid \tau, v, dw) \\
&\quad \left. \times P((\tau, r'), b'(w, y'), \sigma, dy) \right\} P(r'', b''(y''), \sigma, dy') q(\sigma, y) \int_R \\
&\quad \times \pi(y' \mid \sigma, y, dz) P_0(\sigma, z, t, E)
\end{aligned}
$$

with obvious notations r', r'', y', y''. But, this coincides with the right side of (5) by changing the order of integration by $d\sigma$ and $d\tau$, using (3).

Let $\varphi(x)$ be the sum of non-negative functions $\varphi_k(x_k)$, $k=1,2,\cdots$, measurable in $x_k=(x_{i_{1,k}},\cdots,x_{i_{n_k,k}})$, and let $I(x_k)$ be the set of indices for x_k. For a subset J of $I=\{1,2,\cdots\}$, we write

$$
\int f^{\infty}\varphi(x) = \sum_{k=1}^{\infty} \int_{R^{\#(I(x_k))}} \prod_{i \in I(x_k)} f(dx_i)\varphi_k(x_k),
$$

$$
\int_{J^c} f^{\infty}\varphi(x) = \sum_{k=1}^{\infty} \int_{R^{\#(I(x_k)\cap J^c)}} \prod_{i \in I(x_k)\cap J^c} f(dx_i)\varphi_k(x_k).\ [5]
$$

Then, by a similar induction as in II of [3], we have

Theorem 3. *The minimal solution $P^{(f)}(s, x, t, E)$ of (1) is given by*

$$
(6)\quad
\begin{aligned}
P^{(f)}(s, x_1, t, E) &= \int_{\{1\}^c} f^{\infty} \sum_{b \in T} P((s, \cdots, s), b(x), t, E), \\
P_{s,t}^{(f)}(E) &= \int f^{\infty} \sum_{b \in T} P((s, \cdots, s), b(x), t, E).
\end{aligned}
$$

3. Applications. a) *Chapman-Kolmogorov equation*:

$$
(7)\quad P^{(f)}(s, x, u, E) = \int_R P^{(f)}(s, x, t, dy) P^{(P_{s,t}^{(f)})}(t, y, u, E),\ s \le t \le u.
$$

Proof. By (4) and (6), $P^{(f)}(s, x_1, u, E)$ is equal to

$$
\begin{aligned}
&\int_{\{1\}^c} f^{\infty} \sum_{b \in T} \sum_{b' \le b} \int_{R^{\#(b')}} \prod_{b_k \in b/b'} P(s, b_k(x_k), t, dy_k) P(t, b'(y), u, E)^{[6]} \\
&= \int_{\{1\}^c} f^{\infty} \sum_{b' \in T} \int_{R^{\#(b')}} \prod_{k=1}^{\#(b')} \sum_{b_k \in T} P(s, b_k(x_k), t, dy_k) P(t, b'(y), u, E) \\
&= \sum_{b' \in T} \int_{R^{\#(b')}} \left\{ \int_{\{1\}^c} f^{\infty} \sum_{b_1 \in T} P(s, b_1(x_1), t, dy_1) \right\} \prod_{k=2}^{\#(b')} \\
&\quad \times \left\{ \int f^{\infty} \sum_{b_k \in T} P(s, b_k(x_k), t, dy_k) \right\} P(t, b'(y), u, E) \\
&= \int_R P^{(f)}(s, x_1, t, dy_1) \sum_{b' \in T} \int_{R^{\#(b')-1}} \prod_{k=2}^{\#(b')} P_{s,t}^{(f)}(dy_k) P(t, b'(y), u, E)
\end{aligned}
$$

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5) When f is a probability measure, these are the integrals by infinite direct products of f's.

6) $P(s, b(x), t, E)$ is an abbreviation for $P((s, \cdots, s), b(x), t, E)$.

$$=\int_R P^{(f)}(s,x_1,t,dy_1)\int_{\{1\}^c}(P^{(f)}_{s,t})^\infty\sum_{b'\in T}P(t,b'(\boldsymbol{y}),u,E),$$

coinciding with the right side of (7) by (6).

b) *Backward equation* (2) for the minimal solution is proved by rewriting

(8) $$P^{(f)}(s,r,x,t,E)=\int_{\{1\}^c}f^\infty\sum_{b\in T}P((s,r,r,\cdots,r),b(\boldsymbol{x}),t,E),\quad r,s\leq t,$$ [7]

in two ways. First, noting that $b_1\times b_2\rightarrow b_1((b_0,b_2)b_0,\cdots,b_0)$ is a one to one correspondence between $T\times T$ and $T-\{b_0\}$, and using (5), we have

$$\begin{aligned}P^{(f)}(s,r,x_1,t,E)&=\int_{\{1\}^c}f^\infty\Big\{P(s,b_0(x_1),t,E)+\sum_{b\in T-\{b_0\}}\\&\quad\times P((s,r,\cdots,r),b(\boldsymbol{x}),t,E)\Big\}\\&=P_0(s,x_1,t,E)+\int_{\{1\}^c}f^\infty\sum_{b_1\in T}\sum_{b_2\in T}P((s,r,\cdots,r),b_1((b_0(x_1),\\&\quad\times b_2(\boldsymbol{x}'),\boldsymbol{x}''),t,E)\\&=P_0(s,x_1,t,E)+\int_{\{1\}^c}f^\infty\sum_{b_1\in T}\sum_{b_2\in T}\int_{s\vee r}^t d\tau\int_{R^2}P(s,b_0(x_1),\tau,dy)\\&\quad\times P((r,\cdots,r),b_2(\boldsymbol{x}'),\tau,dy')q(\tau,y)\int_R\pi(y'\,|\,\tau,y,dz)\\&\quad\times P((\tau,r,\cdots,r),b_1(z,\boldsymbol{x}''),t,E)\\&=P_0(s,x_1,t,E)+\int_{s\vee r}^t d\tau\int_{R^2}P_0(s,x_1,\tau,dy)P^{(f)}_{r,\tau}(dy')q(\tau,y)\int_R\\&\quad\times\pi(y'\,|\,\tau,y,dz)P^{(f)}(\tau,r,z,t,E).\end{aligned}\tag{9}$$ [8]

On the other hand, we can prove, by (4),

(10) $$P^{(f)}(s,r,x,t,E)=P^{(P^{(f)}_{r,s})}(s,x,t,E),\quad\text{for}\quad r\leq s\leq t,$$ [9]

and hence (2) is obtained by substituting (10) into the left and the right extremes of (9) with r replaced by s_0. In fact, $P^{(f)}(s,r,x,t,E)$ is equal to

$$\int_{\{1\}^c}f^\infty\sum_{b\in T}\sum_{b'\leq b}\int_{R^{\#(b')}}P((s,r,\cdots,r),b_1(\boldsymbol{x}_1),s,dy_1)\prod_{\substack{(b_k\in b/b')\\k\geq 2}}$$
$$P(r,b_k(\boldsymbol{x}_k),s,dy_k)P(s,b'(\boldsymbol{y}),t,E)$$

(where b_1 is the first of b/b')

7) Intuitively, this is the probability that the particle, started at x_1 at time s_1, is in the set E at time t after the interactions governed by b with other particles which started at time r with the common initial distribution f independently. Clearly, this reduces to $P^{(f)}(s,x,t,E)$ when $r=s$.

8) The corresponding equation of forward type is

$$P^{(f)}(s,r,x,t,E)=P_0(s,x,t,E)+\int_{s\vee r}^t d\tau\int_{R^2}P^{(f)}(s,r,x,\tau,dy$$
$$\times P^{(f)}_{r,\tau}(dy')q(\tau,y)\int_R\pi(y'|\tau,y,dz)P_0(\tau,z,t,E).$$

This is proved in a similar way, or by a successive approximation similar to the proof of (6). Note that this reduces to (1) when $r=s$.

9) In case $s\leq r\leq t$, $P^{(f)}(s,r,x,t,E)=\int_R P_0(s,x,r,dy)\,P^{(f)}(r,y,t,E)$.

$$=\int_{\{1\}^c} f^\infty \sum_{b'\in T}\int_{R^{\#(b')}} \sum_{b_1\in T} P((s,r,\cdots,r),b_1(\boldsymbol{x}_1),s,dy_1)\prod_{k=2}^{\#(b')} \times \sum_{b_k\in T} P(r,b_k(\boldsymbol{x}_k),s,dy_k)P(s,b'(\boldsymbol{y}),t,E)$$

$$=\sum_{b'\in T}\int_{R^{\#(b')}} \delta x_1(dy_1)\prod_{k=2}^{\#(b')} P_{r,s}^{(f)}(dy_k)P(s,b'(\boldsymbol{y}),t,E)=P^{(P_{r,s}^{(f)})}(s,x_1,t,E),$$

since $P((s,r,\cdots,r),b(\boldsymbol{x}),s,E)=\delta_{x_1}(E)$ or 0 according as $b=b_0$ or not for $r\le s$.

4. Let $b'\le b$ and define a substochastic measure on $(R^{\#(b')},\mathscr{B}(R^{\#(b')}))$ by

$$P(b/b',\boldsymbol{s},\boldsymbol{x},t,d\boldsymbol{y})=\prod_{b_k\in b/b'} P(s_k,b_k(\boldsymbol{x}_k),t,dy_k),$$

where $\boldsymbol{x}=(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_{\#(b')})$, $\boldsymbol{s}=(s_1,\cdots,s_{\#(b')})$.[10]

Then, the following extension of (4) is proved easily.

$$(4')\qquad P(b/b',\boldsymbol{s},\boldsymbol{x},u,E)=\sum_{b'\le b''\le b}\int_{R^{\#(b'')}} P(b/b'',\boldsymbol{s},\boldsymbol{x},t,d\boldsymbol{y}) \times P(b''/b',t,\boldsymbol{y},u,E)^{11)}$$

Let $b\ge b_1\ge b_2\ge\cdots\ge b_n$, $t_0\le t_1\le\cdots\le t_n$, $E_1\in\mathscr{B}(R^{\#(b_1)}),\cdots,E_n\in\mathscr{B}(R^{\#(b_n)})$, and let

$$P(t_0,t_1,\cdots,t_n;b,b_1,\cdots,b_n;\boldsymbol{x},E_1,\cdots,E_n)=\int_{E_1} P(b/b_1,t_0,\boldsymbol{x},t_1,d\boldsymbol{x}_1)\int_{E_2}$$

$$(11)\qquad \times P(b^1/b_2,t_1,\boldsymbol{x}_1,t_2,d\boldsymbol{x}_2)\cdots\int_{E_{n-1}} P(b_{n-2}/b_{n-1},t_{n-2},\boldsymbol{x}_{n-2},t_{n-1},d\boldsymbol{x}_{n-1}) \times P(b_{n-1}/b_n,t_{n-1},\boldsymbol{x}_{n-1},t_n,E_n).$$

Then, a version of the *consistency condition* holds:

$$P(t_0,t_1,t_3,\cdots,t_n;b_0,b_1,b_3,\cdots,b_n;\boldsymbol{x},E_1,E_3,\cdots,E_n) =\sum_{b_1\ge b_2\ge b_3} P(t_0,t_1,t_2,t_3,\cdots,t_n;b_0,b_1,b_2,\cdots,b_n;\boldsymbol{x},E_1,R^{\#(b_2)}E_3,\cdots,E_n),$$

where we skipped t_2 alone for simplicity. This suggests that (11) is the probability of a cylinder set of a probability space which describes all interactions suffered by the particular particle we are watching at.

References

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10) Since $b/b_0=\{b\}$, $P(b/b_0,s,x,t,E)=P(s,b(x),t,E)$ and (4′) reduces to (4) in case $b'=b_0$.

11) $P(b/b',s,x,t,E)$ is an abbreviation for $P(b/b',(s,\cdots,s),x,t,E)$.