139. *On Radicals of Semigroups with Zero. I*

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The term "semigroup" means in this note always a semigroup with zero element (see [3]). Several concrete types of radicals for semigroups were proposed (see for instance [2], [3], [5]–[9] and [11]). By a ring theoretical analogy (see [4]) also a general theory of radicals for semigroups can be developed.

For any class $\boldsymbol{C}$ of semigroups a $\boldsymbol{C}$-semigroup S means a semigroup belonging to $\boldsymbol{C}$. If a semigroup S has a $\boldsymbol{C}$-ideal $\boldsymbol{C}(S)$ such that $\boldsymbol{C}(S)$ contains any further $\boldsymbol{C}$-ideal of S, then $\boldsymbol{C}(S)$ is called the $\boldsymbol{C}$-radical of S. Semigroups S with $\boldsymbol{C}(S)=0$ are called $\boldsymbol{C}$-semisimple. A class R of semigroups is called *radical*, if the following conditions are satisfied:

1) $\boldsymbol{R}$ is homomorphically closed/not only with respect to forming of Rees factor semigroups/

2) in any semigroup S there exists the $\boldsymbol{R}$-radical $\boldsymbol{R}(S)$

3) the Rees factor semigroup $S/\boldsymbol{R}(S)$ is $\boldsymbol{R}$-semisimple.

The aim of this note is to generalize for semigroups some ring-theoretical results of [1] and [10].

Theorem 1. *For any radical class $\boldsymbol{R}$ of semigroups, and for any ideal $\boldsymbol{J}$ of a semigroup S, the $\boldsymbol{R}$-radical $\boldsymbol{R}(\boldsymbol{J})$ of $\boldsymbol{J}$ is an ideal of S.*

Proof. Assuming that $\boldsymbol{R}(\boldsymbol{J})$ is not an ideal of S, there exists an element $s \in S$ satisfying either $s\boldsymbol{R}(\boldsymbol{J}) \not\subseteq \boldsymbol{R}(\boldsymbol{J})$ or $\boldsymbol{R}(\boldsymbol{J})s \not\subseteq \boldsymbol{R}(\boldsymbol{J})$. If $s\boldsymbol{R}(\boldsymbol{J}) \not\subseteq \boldsymbol{R}(\boldsymbol{J})$, then the union $\boldsymbol{U}=s\boldsymbol{R}(\boldsymbol{J}) \cup \boldsymbol{R}(\boldsymbol{J})$ properly contains $\boldsymbol{R}(\boldsymbol{J})$ and $\boldsymbol{U} \subseteq \boldsymbol{J}$ holds. By $\boldsymbol{J}\boldsymbol{U}=\boldsymbol{J}s\boldsymbol{R}(\boldsymbol{J}) \cup \boldsymbol{J}\boldsymbol{R}(\boldsymbol{J}) \subseteq \boldsymbol{R}(\boldsymbol{J})$ and $\boldsymbol{U}\boldsymbol{J} \subseteq \boldsymbol{U}$ this union $\boldsymbol{U}$ is an ideal of $\boldsymbol{J}$. Being $\boldsymbol{J}/\boldsymbol{R}(\boldsymbol{J})$ $\boldsymbol{R}$-semisimple, $\boldsymbol{U}/\boldsymbol{R}(\boldsymbol{J})$ is not an $\boldsymbol{R}$-semigroup.

By $\varphi_1(r)=sr \cup \boldsymbol{R}(\boldsymbol{J})$ $(r \in \boldsymbol{R}(\boldsymbol{J}))$ is given a mapping of $\boldsymbol{R}(\boldsymbol{J})$ onto $\boldsymbol{U}/\boldsymbol{R}(\boldsymbol{J})$, which by the associativity and

$$\begin{aligned}\varphi_1(r_1, r_2) &= sr_1r_2 \cup \boldsymbol{R}(\boldsymbol{J}) = \boldsymbol{R}(\boldsymbol{J}) \\ &= sr_1s, r_2 \cup \boldsymbol{R}(\boldsymbol{J}) = \varphi_1(r_1), \varphi_1(r_2)\end{aligned}$$

is a homomorphism. Being $\boldsymbol{R}(\boldsymbol{J})$ radical and $\boldsymbol{U}/\boldsymbol{R}(\boldsymbol{J})$ nonradical nonzero semigroups, respectively, this contradiction shows $S\boldsymbol{R}(\boldsymbol{J}) \subseteq \boldsymbol{R}(\boldsymbol{J})$. Similarly can be verified also $\boldsymbol{R}(\boldsymbol{J})S \subseteq \boldsymbol{R}(\boldsymbol{J})$.

Corollary 2. *With the above notations $\boldsymbol{R}(\boldsymbol{J}) \subseteq \boldsymbol{J} \cap \boldsymbol{R}(S)$ holds.*

Proof. $\boldsymbol{R}(\boldsymbol{J})$ is an $\boldsymbol{R}$-ideal of S, contained in $\boldsymbol{R}(S)$.

Corollary 3. *Any ideal of an $\boldsymbol{R}$-semisimple semigroup is again*

$\boldsymbol{R}$-*semisimple*.

Let now $\boldsymbol{C}=\boldsymbol{C}_1$ an arbitrary, homomorphically closed class of semigroups. Assume that for the ordinal numbers $\alpha<\beta$, the classes $\boldsymbol{C}_\alpha$ are already defined. Let $\boldsymbol{C}_\beta$ be the class of all semigroups every non-zero Rees factor semigroup of which contains a non-zero ideal belonging to a class $\boldsymbol{C}_\alpha$ for some $\alpha<\beta$.

Furthermore let $\boldsymbol{LC}_1$ be the union of all classes $\boldsymbol{C}_\alpha$.

A subsemigroup A of S is called accessible, if there exists a finite chain of subsemigroups $A_1, A_2, \cdots, A_{n-1}$ of S such that

$$A=A_0\subset A_1\subset A_2\subset \cdots \subset A_{n-1}\subset A_n=S_1$$

and A_i is an ideal of A_{i+1} for $i=0,1,2,\cdots,n-1$. Such a minimal n is the index of A in S.

Proposition 4. *For any homomorphically closed class* $\boldsymbol{C}=\boldsymbol{C}_1$ *of semigroups, and for the above class* $\boldsymbol{LC}_1$ *any non-zero* $\boldsymbol{LC}_1$*-semigroup* S *has a nonzero accessible subsemigroup* A *contained in* $\boldsymbol{C}_1$.

Proof. Being S a set, by $S\in U\boldsymbol{C}_\alpha$ there is an ordinal number α with $S\in\boldsymbol{C}_\alpha$ for some α. Assume α to be minimal. We prove the existence of a desired A by induction on α. If $\alpha=1$, then $S\in\boldsymbol{C}_1$, and let A be S itself. Assume that our assertion is verified for all semigroups T from $\boldsymbol{C}_\beta$ with $\beta<\alpha$. If $S\in\boldsymbol{C}_\alpha$, and α is minimal, then any non-zero factor semigroup of S, and thus also S itself, contains a non-zero ideal $\boldsymbol{J}$ contained in $\boldsymbol{C}_\beta$ for some $\beta<\alpha$. By induction $\boldsymbol{J}$ has a non-zero accessible subsemigroup A in $\boldsymbol{C}_1$. But A is a non-zero accessible subsemigroup of S, contained in $\boldsymbol{C}_1$.

Proposition 5. *For any non-zero accessible subsemigroup* A *of a semigroup* S *with* $A\in\boldsymbol{C}_1=\boldsymbol{C}$, *the ideal* (A) *of* S *generated by* A_1 *is contained in* $\boldsymbol{C}_n$ *for some finite* n.

Proof. Let n be the index of A in S. We proceed by induction on n. If $n=1$, then $(A)=A\in\boldsymbol{C}_1$. Assume that our proposition is proved for all accessible subsemigroups of index $<n$. Let now A^* be the ideal of A_{n-1} generated by A. By induction A^* lies in $\boldsymbol{C}_{m-1}$ for some finite m, and $A^*\subseteq(A)\subseteq A_{n-1}$. Thus A^* is an ideal of (A), and by $(A)=A\cup AS\cup SA\cup SAS$ we have also $(A)=A^*\cup A^*S\cup SA^*\cup SA^*S$.

We shall prove $(A)\in\boldsymbol{C}_m$. Let $(A)/B$ be any non-zero Rees factor semigroup of (A). If $A^*\not\subseteq B$, then $(A)/B$ with respect to forming of Rees factor semigroups has the nonzero ideal $(A^*\cup B)/B\cong A^*/A^*\cap B$. The righthand side of this isomorphism-relation is a homomorphic image of A^* from the class $\boldsymbol{C}_{m-1}$, and thus $A^*/A^*\wedge B\in\boldsymbol{C}_{m-1}$, being every $\boldsymbol{C}_\alpha$ closed. Then $(A)/B$ has a non-zero ideal in $\boldsymbol{C}_{m-1}$.

Consequently we assume $A^*\subseteq B$. If $A^*S\not\subseteq B$, then there is an element $s\in S$ such that $A^*s\not\subseteq B$. It is easy to prove that $(A^*s\cup B)/B$ is a non-zero twosided ideal of $(A)/B$. Furthermore $\varphi_2 a=as\cup B (a\in A^*)$

maps A^* onto the factor-semigroup $(A^*s \cup B)/B$, and this mapping φ_2 is by

$$\varphi_2 a_1 a_2 = a_1 a_2 s \cup B \subseteq A^*A^*S \cup B \subseteq A^*(A) \cup B \subseteq B$$

and by

$$\varphi_2 a_1 \cdot \varphi_2 a_2 \subseteq A^*SA^*S \cup B \subseteq A^*(A) \cup B \subseteq A^* \cup B = B$$

a homomorphism. Consequently $(A^*s \cup B)/B$ is a non-zero ideal of $(A)/B$ in $\boldsymbol{C}_{m-1}$. Similarly it can be shown that $(sA^* \cup B)/B$ is a non-zero ideal of $(A)/B$, contained in C_{m-1}, for some $s \in S$, if $SA^* \not\subseteq B$.

In what follows, we assume $A^* \cup SA^* \cup A^*S \subseteq B$. By $(A) \neq B$ holds $SA^*S \not\subseteq B$, and thus there exist elements $s \in S_1 t \in S$ such that $sA^*t \not\subseteq B$. It can be verified, that $(sA^*t \cup B)/B$ is an ideal of $(A)/B$. Furthermore by $\varphi_3 a = sat \cup B$ $(a \in A^*)$ the semigroup A^* is homomorphically mapped onto $(sA^*t \cup B)/B$, being $\varphi_3 a_1 a_2 = B = \varphi_3 a_1 \cdot \varphi_3 a_2$. Consequently (A) lies in a class $\boldsymbol{C}_m$ for some finite m.

Theorem 6. *The class $\boldsymbol{LC}_1$, determined by any homomorphically closed class $\boldsymbol{C}_1 = \boldsymbol{C}$ of semigroups, coincides with the class $\boldsymbol{C}_{\omega_0}$, where ω_0 is the first infinite ordinal number.*

Proof. Obviously $\boldsymbol{C}_{\omega_0} \subseteq \boldsymbol{LC}_1$ holds. Furthermore, if S is any nonzero semigroup from $\boldsymbol{LC}_1$, then any non-zero Rees factorsemigroup S/T of S lies in $\boldsymbol{LC}_1$, and S/T contains by Proposition 4 a nonzero accessible subsemigroup A, lying in $\boldsymbol{C}_1$. By Proposition 5 S/T contains a nonzero ideal $(A)/T$ contained in $\boldsymbol{C}_m$ for a finite m. Consequently $\boldsymbol{LC}_1 = \boldsymbol{C}_{\omega_0}$.

Theorem 7. *If $\boldsymbol{C}_1$ is a homomorphically closed and hereditary class of semigroups (i.e. any ideal of a semigroup from $\boldsymbol{C}_1$ lies again in $\boldsymbol{C}_1$), containing all nilpotent semigroups, then $\boldsymbol{LC}_1 = \boldsymbol{C}_2$ holds.*

Proof. If S is any semigroup from $\boldsymbol{C}_3$, then any non-zero factor semigroup $S^1 = S/T$ of S contains a nonzero ideal $\boldsymbol{J}$ belonging to $\boldsymbol{C}_2$. Thus also $\boldsymbol{J}$ contains a nonzero ideal $\boldsymbol{K}$ lying in $\boldsymbol{C}_1$. If $(\boldsymbol{K})$ is the ideal of S' generated by $\boldsymbol{K}$, then $\boldsymbol{K} \subseteq (\boldsymbol{K}) \subseteq \boldsymbol{J}$. Being

$$(\boldsymbol{K})^3 \subseteq \boldsymbol{J}(\boldsymbol{K} \cup \boldsymbol{K}S' \cup S'\boldsymbol{K} \cup S'\boldsymbol{K}S')\boldsymbol{J} \subseteq \boldsymbol{K}$$

and $\boldsymbol{C}_1$ a hereditary class, one has $(\boldsymbol{K})^3 \in \boldsymbol{C}_1$ and $(\boldsymbol{K}) \neq 0$. If $(\boldsymbol{K})^3 \neq 0$, then it is a nonzero $\boldsymbol{C}_1$-ideal of S'. In the case $(\boldsymbol{K})^3 = 0$ and $(\boldsymbol{K})^2 \neq 0$ obviously $(\boldsymbol{K})^2 \in \boldsymbol{C}_1$ being $(\boldsymbol{K})^2$ nilpotent. If $(\boldsymbol{K})^2 = 0$, then also $(\boldsymbol{K}) \in \boldsymbol{C}_1$, and thus S' has a nonzero $\boldsymbol{C}_1$-ideal. Consequently $\boldsymbol{LC}_1 = \boldsymbol{C}_2$.

It would be interesting to investigate that in what case $\boldsymbol{LC}_1$ is a radical class for a homomorphically closed class $\boldsymbol{C}_1$ of semigroups. (A homomorphism generally cannot be given by a Rees factor semigroup.)

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