182. *Local Comparison Theorems for a Certain Class of Multi-Dimensional Markov Processes of Transient Type*

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Our aim of this paper is to give certain local comparison theorems about hitting probability and fine topology in connection with Green functions. In the author's preceding papers [1], [2] and [3] we have given some comparison theorems on fine topology in case Markov processes have Green functions with a certain kind of isotropic singularities. Using the following results of this paper we can get a local comparison theorem about fine topology without the assumption on the singularity of Green functions, or without assuming the existence of Green functions. But our result may not contain the result of Theorem 5 in [2] completely if we restrict our processes to those in Theorem 5 in [2].

1. Notations and definitions. Without special mentioning the process $X=(x_t, \zeta, M_t, P_x)$ we treat is assumed to be a standard process on a domain $\Omega\subset R^d$ (d-dimentional Euclidean space) such that each point in Ω can not be reached[1] and G^{α}[2] maps $C_K(\Omega)$ into $C_0(\Omega)$ for some $\alpha>0$, where $C_0(\Omega)$ denotes the set of continuous functions on Ω vanishing at infinity and $C_K(\Omega)$ is the set of functions $\in C_0(\Omega)$ with compact support in Ω, and $\int_0^{+\infty} T_t f dt$ is finite for each $f\in C_K(\Omega)$.

Definition 1. A non-negative, measurable kernel $G(x,y)$ on $\Omega\times\Omega$ is said to be a regular Green function of the process X, if it satisfies the following conditions.

i) $\int_0^{+\infty} T_t f(x)dt=\int_\Omega G(x,y)f(y)dy$ for each $f\in C_K(\Omega)$.

ii) $G(x,y)$ is bounded except at each neighborhood of the diagonal on $\Omega\times\Omega$.

iii) $\lim_{n\to+\infty} \inf_{\substack{x\in Q_n\\ y\in Q_n}} G(x,y)=+\infty$ for each ball decreasing to a point.

iv) For each compact or open set K with compact closure in Ω, there exists a measure $\mu_K(dy)$ supporting on $\bar{K}$ such that

$$P_x(\sigma_K<+\infty)=\int G(x,y)\mu_K(dy).$$

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1) $P_x(\sigma\{x_0\}<+\infty)=0$ for each point $x, x_0\in\Omega$, where $\sigma_B=\inf(t>0, x_t\in B)$.

2) $G^\alpha f(x)=\int_0^{+\infty} e^{-\alpha t}T_t f(x)dt$ for $f\in C_K(\Omega)$.

For convenience we shall introduce some notations. The first one is suggested by the paper [4] by E. M. Landis.

Definition 2. A measurable function f on Ω is said to be C-weakly subharmonic (resp. C-weakly superharmonic) at (x_0, Q) with respect to a process X, where Q is an open set in Ω containing x_0, if for each open set S such that $Q \supset \bar{S} \supset S \ni x_0$ it holds

$$E_{x_0} f(x_{\tau_S}) \geqq C \cdot f(x_0)$$

(resp. $E_{x_0} f(x_{\tau_S}) \leqq C \cdot f(x_0)$),
where $\tau_S = \inf(t \geqq 0, x_t \in \Omega - S)$.

Definition 3. Let $X_1 = (x_t^1, \zeta^1, M_t^1, P_x^1)$ and $X_2 = (x_t^2, \zeta^2, M_t^2, P_x^2)$ be processes on Ω_1 and Ω_2 respectively and Q be an open set such that $x_0 \in Q \subset \Omega_1 \cap \Omega_2$. We say that X_1 is C-dominated by X_2 at (x_0, Q), if for each compact or open set K with compact closure in Q

$$P_{x_0}^1(\sigma_K < +\infty) \leqq C \cdot P_{x_0}^2(\sigma_K < +\infty).$$

If X_1 is C-dominated by X_2 and X_2 is C'-dominated by X_1 at (x_0, Q), we say that X_1 and X_2 are (C, C')-dominated at (x_0, Q) each other. If for each point in $\Omega_1 \cap \Omega_2$ there exists a neighborhood Q such that for each point $x_0 \in Q$ X_1 and X_2 are (C, C')-dominated each other at (x_0, Q), where C and C' are positive constants independent of the choice of $x_0 \in Q$, we say that X_1 and X_2 are locally dominated each other in $\Omega_1 \cup \Omega_2$.

Definition 4. Let X_1 and X_2 be the processes on Ω_1 and Ω_2 having Green functions $G_1(x, y)$ and $G_2(x, y)$ respectively. We say that G_1 and G_2 have the same local singularity, if for each point in $\Omega_1 \cap \Omega_2$ there exists a neighborhood Q and constants $C_1 \geqq C_2 > 0$ depending only on the choice of Q such that

$$C_2 G_2(x, y) \leqq G_1(x, y) \leqq C_1 G_2(x, y) \qquad (x, y) \in Q \times Q.$$

2. Theorems. Let X be a process on Ω and K be a Borel set in Ω. K_X^{reg} denotes the set of all regular points of K with respect to X, that is,

$$K_X^{\text{reg}} = \{x ; P_x(\sigma_K = 0) = 1\}$$

Theorem 1. *Let $X_1 = (x_t^1, \zeta^1, M_t^1, P_x^1)$ and $X_2 = (x_t^2, \zeta^2, M_t^2, P_x^2)$ be processes having the properties in § 1 on Ω_1 and Ω_2 respectively and let us assume that X_1 has a regular Green function $G_1(x, y)$. Then, for some point $x_0 \in \Omega_1 \cap \Omega_2$ and a ball $Q_{x_0} = \{x ; |x - x_0| < T_0\} \subset \Omega_1 \cap \Omega_2$, if a function $G_1(\cdot, y)$[3] is C-weakly subharmonic (resp. C-weakly superharmonic) at $(x_0, Q_{x_0} - \{y\})$ with respect to X_2 for each $y \in Q_{x_0}$, where C is a positive constant independent of the choice of $y \in Q_{x_0}$, $x_0 \in K_{X_1}^{\text{reg}}$ implies $x_0 \in K_{X_2}^{\text{reg}}$ (resp. $x_0 \in K_{X_2}^{\text{reg}}$ implies $x_0 \in K_{X_1}^{\text{reg}}$), and further X_1 is C_1-dominated by X_2 at $(x_0, \tilde{Q}_{x_0})$ for some ball $\tilde{Q}_{x_0}$ centering at x_0, where C_1 is a positive constant depending on the choice of a ball $\tilde{Q}_{x_0}$. (resp. X_2*

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3) Hereafter without special mentioning a function f related to the process X on Ω is regarded as a function on R^d such that $f(x) = 0$ for $x \in \Omega^c$.

is C_2-dominated by X_1 at $(x_0, \tilde{Q}_{x_0})$, where C_2 is a positive constant depending on the choice of a ball $\tilde{Q}_{x_0}$ under an additional assumption;

$$\lim_{n\to+\infty} \sup_{x\in Q} {}_c P_x^2(\sigma_{Q_n}^2 < +\infty)=0$$

holds for each non-empty open set $Q\subset \Omega_2$ and a sequence of open sets $\{Qn\}$ converging to a point in Q.)

The following two theorems can be proved using Theorem 1.

Theorem 2. *Let X_1 and X_2 be the processes on Ω_1 and Ω_2 having regular Green functions $G_1(x, y)$ and $G_2(x, y)$ respectively. If G_1 and G_2 have the same local singularity on $\Omega_1\cap\Omega_2$, X_1 and X_2 are locally dominated each other in $\Omega_1\cap\Omega_2$ and the fine topologies introduced by X_1 and X_2 respectively coincides on $\Omega_1\cap\Omega_2$.*

Theorem 3. *Let X_1 and X_2 be processes in Theorem 2 and let us assume $\Omega_1=\Omega_2=\Omega$ and $G_1(x, y)$ and $G_2(x, y)$ are continuous on $\Omega\times\Omega$ except at the diagonal. Then the following three conditions are equivalent.*

I) *G_1 and G_2 have the same local singularity on Ω.*

II) *X_1 and X_2 are locally dominated each other on Ω.*

III) *For each point in Ω there exists a neighborhood Q such that $G_1^y(\cdot)$[4] (resp. $G_2^y(\cdot)$) is C-weakly superharmonic at $(x, Q-\{y\})$ for each $x, y\in Q$ with respect to X_2 (resp. X_1), where C is a positive constant which does not depend on the choice of $x, y\in Q$ but may depend on the choice of Q.*

3. A converse theorem on fine topology for isotropic Lévy processes. In the preceding paper [3] we have given a converse theorem on fine topology. Here we give such one for the isotropic Lévy process which is a slight generalization of the Corollary in [3].

Theorem 4. *If a isotropic Lévy process on R^d $(d=3)$ and α-symmetric stable process $(\alpha>1)$on R^d introduce the same fine topology, the isotropic Lévy process has a regular Green function which has the local singularity $r\alpha-d$.*

References

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[2] ——: On the singularity of Green functions in Markov processes. Nagoya Math. J., **33**, 21–52 (1968).

[3] ——: On the singularity of Green functions in Markov processes. II. Nagoya Math. J., **37**, 209–217 (1970).

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4) $G_i^y(\cdot)=G_i(\cdot, y)$, $i=1,2$.