180. *Complex Powers of Non-elliptic Operators*

By Michihiro NAGASE*) and Kenzo SHINKAI**)

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1970)

1. Introduction.

In the present paper we shall construct symbols of pseudo-differential operators which define complex powers of a pseudo-differential operator in a class S_λ^m which contains semi-elliptic operators. Complex powers of an elliptic operator as pseudo-differential operators are defined by Burak [1] and Seely [4]. They constructed symbols through Dunford's integrals for an elliptic operator defined on a C^∞ compact manifold without boundary, so the global ellipticity of the operator is required. Here, we shall construct symbols only by local calculation. The precise calculation of symbols for iterations of a pseudo-differential operator gives the relations among polynomials in coefficients of the symbols, then the symbols of integral powers of an operator is extended to be those of complex ones.

The authors wish to thank Prof. H. Kumano-go for suggesting this problem and the proof of Lemma 3.1.

2. Definitions and lemmas.

Definition 1. *A real valued $C^\infty(R^n)$ function $\lambda(\xi)$ is called a basic weight function when it satisfies the conditions*:

(2.1) $$1 \leqq \lambda(\xi) \leqq A(1+|\xi|),$$

(2.2) $$|\partial_\xi^\alpha \lambda(\xi)| \leqq A_\alpha \lambda(\xi)^{1-|\alpha|} \quad \textit{for any} \quad \alpha,$$

for some constants A and A_α. (*See Kumano-go* [3].)

Definition 2. *Let $\lambda(\xi)$ be a basic weight function. Then we say $p(x,\xi) \in S_\lambda^m$, when $p(x,\xi) \in C^\infty(R^n \times R^n)$ and*

(2.3) $$|D_x^\alpha \partial_\xi^\beta p(x,\xi)| \leqq C_{\alpha,\beta} \lambda(\xi)^{m-|\beta|} \quad \textit{for any} \quad \alpha, \beta,$$

for some constants $C_{\alpha,\beta}$, where $D_x = (-i)\partial_x$.

For $p(x,\xi) \in S_\lambda^m$ we define the pseudo-differential operator $p(X, D_x)$ by

(2.4) $$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} p(x,\xi) \hat{u}(\xi) d\xi,$$

where $u(x)$ is a C^∞ function which together with all their derivatives decreases faster than any powers of $|x|$ as $|x| \to \infty$, and

$$\hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x) dx.$$

We denote the symbol of an operator $p(X, D_x)$ by

---

*) Osaka Industrial University.

**) University of Osaka Prefecture.

$$p(x, \xi)=\sigma(p(X, D_x)).$$

For two operators $p(X, D_x)$ and $q(X, D_x)$,

$$p(X, D_x)\equiv q(X, D_x) \text{ means } \sigma(p(X, D_x))-\sigma(q(X, D_x)) \in S_\lambda^{-\infty},$$

where $S_\lambda^{-\infty}=\bigcap_{-\infty<m<\infty} S_\lambda^m$.

In what follows a basic weight function $\lambda(\xi)$ satisfies the condition

$$C_0(1+|\xi|)^\rho \leqq \lambda(\xi) \qquad (0<\rho\leqq 1), \tag{2.5}$$

for some constant $C_0>0$.

Lemma 2.1 (*Hörmander* [2]). *If $p_j(x, \xi) \in S^{m_j}$, $j=0, 1, \cdots$ and $m_0>m_1>m_2>\cdots\to-\infty$, there exists $p(x, \xi) \in S_\lambda^{m_0}$ such that*

$$p(x, \xi)-\sum_{j=0}^{N} p_j(x, \xi) \in S_\lambda^{m_{N+1}}$$

and $p(x, \xi)$ is uniquely determined modulo $S_\lambda^{-\infty}$.

Lemma 2.2 (*Hörmander* [2]). *If $p_j(x, \xi) \in s_\lambda^{m_j}$ $(j=1,2)$,*

$$\sigma(p_1(X, D_x)p_2(X, D_x))=\sum_{|\alpha|<N} \frac{1}{\alpha\,!} p_1^{(\alpha)}(x, \xi)p_{2(\alpha)}(x, \xi)+r_N(x, \xi) \tag{2.6}$$

where $\qquad r_N(x, \xi) \in S_\lambda^{m_1+m_2-N}$.

Definition 3. *We say that $p(x, \xi)$ is λ-elliptic of order m if $p(x, \xi) \in S_\lambda^m$ and*

$$p(x, \xi)\geqq \delta\lambda(\xi)^m \qquad (\delta>0). \tag{2.7}$$

3. Complex powers of λ-elliptic operators.

Theorem 1. *Let $p(x, \xi) \in S_\lambda^m$ be λ-elliptic and $Arg\, p(x, \xi) \neq \pi$. Then there exists a family $\{p(z\,; X, D_x)\}$ of operators with parameter $z \in C$ which satisfies the following conditions:*

$$p(z_1\,; X, D_x)\cdot p(z_2\,; X, D_x)\equiv p(z_1+z_2\,; X, D_x), \tag{3.1}$$

$p(1\,; X, D_x)\equiv p(X, D_x)$, $p(0\,; X, D_x)\equiv I$ (the identity operator),

(3.2) *$p(z\,; x, \xi)$ is an entire function of z,*

$$p(z\,; x, \xi)-p(x, \xi)^z \in S_\lambda^{mRez-1}. \tag{3.3}$$

Proof. When z is equal to a positive integer l, using Lemma 2.2 repeatedly, we have

$$\begin{aligned}\sigma(\{p(X, D_x)\}^l)=\sigma(\overbrace{p(X, D_x)\cdots p(X, D_x)}^{l}) \\ \times p(x, \xi)^l+\sum_{j=1}^{N-1} p_j(l\,; x, \xi)+r_N(l\,; x, \xi)\end{aligned} \tag{3.4}$$

where $p_j(l\,; x, \xi) \in S_\lambda^{ml-j}$, $r_N(l\,; x, \xi) \in S_\lambda^{ml-N}$ and

$$\begin{aligned}p_j(l\,; x, \xi)=\sum \frac{1}{\alpha_2^1!\alpha_3^1!\cdots\alpha_l^{l-1}!} p^{(\beta_1)}(x, \xi)p^{(\beta_2)}_{(\alpha_2^1)}(x, \xi) \\ \times p^{(\beta_3)}_{(\alpha_3^1+\alpha_3^2)}(x, \xi)\cdots p^{(\beta_{l-1})}_{(\alpha_{l-1}^1+\cdots+\alpha_{l-1}^{l-2})}(x, \xi)p_{(\alpha_l^1+\cdots+\alpha_l^{l-1})}(x, \xi)\end{aligned} \tag{3.5}$$

where $p^{(\beta)}_{(\alpha)}(x, \xi)=D_x^\alpha\partial_\xi^\beta p(x, \xi)$ and

$$\begin{aligned}\beta_1&=\alpha_2^1+\alpha_3^1+\cdots+\alpha_l^1\\ \beta_2&=\qquad\ \ \alpha_3^2+\cdots+\alpha_l^2\\ &\cdots\cdots\\ \beta_{l-1}&=\qquad\qquad\qquad\ \alpha_l^{l-1}\end{aligned}$$

The summation in the right hand side of (3.5) is taken for all the sets $(\alpha_2^1, \alpha_3^1, \alpha_3^2, \cdots, \alpha_l^{l-1})$ of multi-indices α_i^k which satisfy the condition $|\beta_1+\beta_2+\cdots+\beta_{l-1}|=j$.

Now, we can write

$$(3.6)\qquad p_j(l; x, \xi)=\sum_{k=2}^{2j} \frac{l(l-1)\cdots(l-k+1)}{k!} p(x, \xi)^{l-k} p_{j,k}(x, \xi)$$

where $p_{j,k}(x, \xi) \in S_\lambda^{km-j}$ and

$$(3.7)\qquad p_{j,k}(x, \xi)=\sum \frac{1}{\alpha_2^1!\alpha_3^1!\cdots\alpha_k^{k-1}!} p^{(\beta_1)}(x, \xi) p^{(\beta_2)}_{(\alpha_2^1)}(x, \xi)$$

$$\times p^{(\beta_3)}_{(\alpha_3^1+\alpha_3^2)}(x, \xi)\cdots p^{(\beta_{k-1})}_{(\alpha_{k-1}^1+\cdots+\alpha_{k-1}^{k-2})}(x, \xi) p_{(\alpha_k^1+\cdots+\alpha_k^{k-1})}(x, \xi)$$

$$\beta_i=\alpha_{i+1}^i+\cdots+\alpha_k^i \qquad (i=1, \cdots, k-1)$$

and the summation in (3.7) is taken for

$$|\beta_1+\cdots+\beta_{k-1}|=j \qquad |\beta_1|\neq 0$$

$$|\beta_i|+|\alpha_i^1+\cdots+\alpha_i^{i-1}|\neq 0 \qquad (i=2, \cdots, k-1)$$

and $\qquad |\alpha_k^1+\cdots+\alpha_k^{k-1}|\neq 0.$

Here we note that $p_{j,k}(x, \xi)$ are independent of l.

Since $p(x, \xi)$ is λ-elliptic and $\operatorname{Arg} p(x, \xi)\neq\pi$ by Lemma 2.1 and (3.6) we can define

$$(3.8)\qquad p(z; x, \xi)\sim p(x, \xi)^z+\sum_{j=1} p_j(z; x, \xi)$$

Since $\{p(X, D_x)\}^{l_1+l_2}=\{p(X, D_x)\}^{l_1}\cdot\{p(X, D_x)\}^{l_2}$ for any positive integers l_1, l_2, by Lemma 2.2 and (3.4) we have

$$(3.9)\qquad p_j(l_1+l_2; x, \xi)=\sum_{\substack{j_1+j_2+|\alpha|=j\\ j_1\geq 0, j_2\geq 0}} \frac{1}{\alpha!} p_{j_1}^{(\alpha)}(l_1; x, \xi) p_{j_2(\alpha)}(l_2, x, \xi),$$

where $p_0(l; x, \xi)=p(x, \xi)^l$.

Then we have

$$(3.10)\qquad p(l_1+l_2; X, D_x)\equiv p(l_1; X, D_x)\cdot p(l_2; X, D_x).$$

Since $p(x, \xi)$ is λ-elliptic, we can divide both sides of (3.9) by $p(x, \xi)^{l_1+l_2}$, and using (3.6) both sides become polynomials of l_1 and l_2. Then (3.9) holds even if l_1 and l_2 are replaced with any complex numbers. Then we have

$$(3.10)'\qquad p(z_1+z_2; X, D_x)\equiv p(z_1; X, D_x)\cdot p(z_2; X, D_x)$$

for any $z_1, z_2 \in C$.

Thus we have (3.1) and (3.3). By checking the proof of Lemma 2.1 in Hörmander [2], we have (3.2).

Lemma 3.1. *Let $p(x, \xi) \in S_\lambda^m$ satisfy the assumptions in Theorem 1 and let $\{p^{(1)}(z; X, D_x)\}$, $\{p^{(2)}(z; X, D_x)\}$ satisfy the conditions* (3.1) *and* (3.3). *Then for any positive rational number r,*

$$(3.11)\qquad p^{(1)}(r; x, \xi)-p^{(2)}(r; x, \xi) \in S_\lambda^{-\infty}.$$

Proof. Let $p^{(2)}\left(\frac{k}{l}; x, \xi\right)=p^{(1)}\left(\frac{k}{l}; x, \xi\right)+r\left(\frac{k}{l}; x, \xi\right)$. Then by (3.3), $r\left(\frac{k}{l}; x, \xi\right) \in S_\lambda^{mk/l-1}$.

Now, if we assume $r\left(\frac{k}{l}; x, \xi\right) \in S_\lambda^{mk/l-\nu}$, $\nu=1,2,\cdots$, then we have $r\left(\frac{k}{l}; x, \xi\right) \in S_\lambda^{mk/l-(\nu+1)}$.

Indeed, by (3.1),

$$\begin{aligned}0&\equiv \left\{p^{(2)}\left(\frac{k}{l}; X, D_x\right)\right\}^l - \left\{p^{(1)}\left(\frac{k}{l}; X, D_x\right)\right\}^l \\ &= \left\{p^{(1)}\left(\frac{k}{l}; X, D_x\right)\right\} + r\left(\frac{k}{l}; X, D_x\right)\Big\}^l - \left\{p^{(1)}\left(\frac{k}{l}; X, D_x\right)\right\}^l \\ &= \sum_{j=1}^{l} \left\{p^{(1)}\left(\frac{k}{l}; X, D_x\right)\right\}^{l-j} \cdot r\left(\frac{k}{l}; X, D_x\right) \cdot \left\{p^{(1)}\left(\frac{k}{l}; X, D_x\right)\right\}^{j-1} \\ &\quad + q\left(\frac{k}{l}; X, D_x\right)\end{aligned}$$

where $\sigma\left(q\left(\frac{k}{l}; X, D_x\right)\right) \in S_\lambda^{km-2}$, thus we have

$$\sigma\left(r\left(\frac{k}{l}; X, D_x\right)\right) \in S_\lambda^{k/lm-(\nu+1)}.$$

Hence we have the lemma.

Theorem 2. *Let $p(x, \xi) \in S_\lambda^m$ satisfy the assumptions in Theorem 1. If $\{p^{(i)}(z; X, D_x)\}$ $(i=1.2)$ satisfy (3.1), (3.2),(3.3) and $\sigma(p^{(i)}(z; X, D_x))$ $\sim p(x, \xi)^z \left\{1 + \sum_{j=1}^{\infty} \sum_{k=1}^{N_j^{(i)}} C_{j,k}^{(i)}(z) p_{j,k}^{(i)}(x, \xi)\right\}$ where $p_{j,k}^{(i)}(x, \xi) \in S_\lambda^{-j}$.*

Then, $p^{(1)}(z; X, D_x) \equiv p^{(2)}(z; X, D_x)$.

Proof. By Lemma 3.1 we get $p^{(1)}(r; X, D_x) \equiv p^{(2)}(r; X, D_x)$ for any positive rational number r. This means

$$\sum_{j=1}^{N} \left\{\sum_{k=1}^{N_j^{(1)}} C_{j,k}^{(1)}(r) p_{j,k}^{(1)}(x, \xi) - \sum_{k=1}^{N_j^{(2)}} C_{j,k}^{(2)}(r) p_{j,k}^{(2)}(x, \xi)\right\} \in S_\lambda^{-N-1} \tag{3.12}$$

for any N. Now we show

$$\sum_{j=1}^{N} \left\{\sum_{k=1}^{N_j^{(1)}} C_{j,k}^{(1)}(z) p_{j,k}^{(1)}(x, \xi) - \sum_{k=1}^{N_j^{(2)}} C_{j,k}^{(2)}(z) p_{j,k}^{(2)}(x, \xi)\right\} \in S_\lambda^{-N-1} \tag{3.12$'$}$$

for any z.

For $N=1$, we can write

$$\sum_{k=1}^{N_1^{(1)}} C_{1,k}^{(1)}(z) p_{1,k}^{(1)}(x, \xi) - \sum_{k=1}^{N_1^{(2)}} C_{1,k}^{(2)}(z) p_{1,k}^{(2)}(x, \xi) = \sum_{k=1}^{M} C_{1,k}(z) p_{1,k}(x, \xi)$$

where $C_{1,k}(z)$ are linearly independent. Since $C_{1,k}(z)$ are analytic, there exist positive rational numbers r_j such that $(C_{1,1}(r_j), \cdots, C_{1,M}(r_j))$ $j=1, \cdots, M$ are linearly independent as M-vectors, and by (3.12), $\sum_{k=1}^{M} C_{1,k}(r_j) p_{1,k}(x, \xi) \in S_\lambda^{-2}$ $j=1, \cdots, M$. Hence $p_{1,k}(x, \xi)$ are linear combinations of $p(r_j; x, \xi) \in S_\lambda^{-2}$. This means (3.12)$'$ holds for $N=1$.

Using the same method we can prove (3.12)$'$ by induction.

Q.E.D.

Remark 1. The assumptions for $p(x,\xi)$ in Theorem 1 can be weakened as follows:

$$\operatorname{Arg} p(x,\xi) \neq \pi, \quad \text{for} \quad |\xi| \geqq C,$$
$$p(x,\xi) \geqq \delta\lambda(\xi)^m \quad \text{for} \quad |\xi| \geqq C,\ \delta > 0.$$

Remark 2. If the assumptions for $p(x,\xi)$ in Theorem 1 are satisfied only for $x \in \Omega$, where Ω is an open set, then $\{p(z;x,\xi)\}$ can also be constructed in the same Ω.

Remark 3. For a system of pseudo-differential operators similar results are obtained by K. Hayakawa and H. Kumano-go [5].

References

[1] T. Burak: Fractional powers of elliptic differential operators. Ann. Scoula Norm. Sup. Pisa, **22**, 113–132 (1968).

[2] L. Hörmander: Pseudo-differential operators and hypoelliptic equations. Proc. Symposium on Singular Integrals. Amer. Math. Soc., **10**, 138–183 (1967).

[3] H. Kumano-go: Pseudo-differential operators and the uniqueness of the Cauchy problem. Comm. Pure Appl. Math., **22**, 73–129 (1969).

[4] R. T. Seely: Complex powers of an elliptic operator. Proc. Symposium on Singular Integrals. Amer. Math. Soc., **10**, 288–307 (1967).

[5] K. Hayakawa and H. Kumano-go: Complex powers of a system of pseudo-differential operators (to appear).