179. On Some Invariant Subspaces

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Let X be a compact Hausdorff space and let A be a function algebra on X. Throughout this paper, ϕ will be a fixed multiplicative linear functional on A which admits a unique representing measure m. Further we assume that the Gleason part of ϕ is non trivial. We denote by A_0 the maximal ideal associated with ϕ; $A_0=\{f\in A:\phi(f)=0\}$. Let $H^2=H^2(dm)$ be the closure $[A]_2$ of A in $L^2=L^2(dm)$. We put $H_0^2=\left\{f\in H^2;\int f dm=0\right\}$. We shall refer to Browder [1] for the abstract function theory in this situation.

Let M be a closed subspace of H^2. M is called simply invariant if $[A_0M]_2 \subsetneqq M$. We call M complementary invariant if $H^2\ominus M$, the orthogonal complement of M in H^2, is simply invariant. The purpose of this paper is a characterization of the complementary invariant subspace.

It is well known that L^2 admits the orthogonal decomposition $L^2=H^2\oplus\bar{H}_0^2$, where the bar denotes the complex conjugation. We denote by P the orthogonal projection of L^2 onto H^2 As Wermer has shown, there exists an inner function Z such that $H_0^2=ZH^2$. (See [1] Lemma 4.4.3 for our situation.) We define the backward shift operator T on H^2 by

$$Tf=\frac{f-\int f dm}{Z} \quad (f\in H^2).$$

Theorem. *The complementary invariant subspaces of H^2 are precisely the subspaces of the form*

$$P[Tq\cdot\bar{H}^2],$$

where q is an inner function. q is determined by the subspace up to a constant factor.

Proof. Let M be a complementary invariant subspace of H^2. Then $N=H^2\ominus M$ is a simply invariant subspace of H^2. Therefore, by the generalized Beurling theorem (for instance, see [1] Theorem 4.3.5), N has the form $N=qH^2$, where q is inner. For simplicity, we put $h=Tq$. Evidently $h\in L^\infty\cap H^2$. Since $\int Z dm=0$ and q is inner, we have

$$(h, qf)=\left(\frac{q-\int q dm}{Z}, qf\right)=(1, Zf)-\int q dm\cdot(1, Zqf)=0^{*)}$$

for every $f\in H^2$. Thus $h\perp qH^2=N$. Hence $h\in M=H^2\ominus N$. We next show that $M\supset P[h\cdot\bar{H}^2]$. Let $f\in N$. Since N is A-invariant and $h\in M$, we have $(f, h\bar{g})=(gf, h)=0$ for all $g\in A$. Thus $(f, P(h\bar{g}))=(f, h\bar{g})=0$ for all $g\in H^2$. Hence $N\subset H^2\ominus P[h\cdot\bar{H}^2]$ and so $M\supset P[h\cdot\bar{H}^2]$. Let now $f\in M\ominus P[h\cdot\bar{H}^2]$. Then, for all $g\in H^2$, we have

$$0=(P(h\bar{g}), f)=(h\bar{g}, f)=(\bar{f}h, g)$$
$$=\left(\bar{f}\frac{q-\int q dm}{Z}, g\right)=\left(\frac{q}{Z}\bar{f}, g\right)-\int q dm\cdot\int\overline{g f Z}dm=\left(\frac{q}{Z}\bar{f}, g\right).$$

Thus $\frac{q}{Z}\bar{f}\perp H^2$, and $\frac{q}{Z}\bar{f}\in\bar{H}^2_0$, so $\frac{Z}{q}f\in H^2_0$. Therefore $f\in\frac{q}{Z}H^2_0=qH^2=N$. But $f\in M\perp N$. Hence $f=0$ a.e., so $M=P[h\cdot\bar{H}^2]$.

Conversely, suppose that $M=P[Tq\cdot\bar{H}^2]$ for some inner function q. We show that M is complementary invariant. By the generalized Beurling theorem, it suffices to see that $H^2\ominus M$ has the form $q\cdot H^2$. Clearly $qH^2\subset H^2$. If $f\in H^2$, then

$$(qf, P(Tq\cdot\bar{g}))=(qf, Tq\cdot\bar{g})=\left(qf, \frac{q-\int q dm}{Z}\bar{g}\right)$$
$$=(f, \bar{Z}\bar{g})-\int\bar{q}dm(qf, \bar{Z}\bar{g})=0 \quad (\forall g\in H^2).$$

Hence $qH^2\subset H^2\ominus M$. Next, suppose that $f\in\{H^2\ominus M\}\ominus qH^2$. Since $f\perp M$, we have

$$0=(f, P(Tq\cdot\bar{g}))=(f, Tq\cdot\bar{g})=\left(f, \frac{q-\int q dm}{Z}\bar{g}\right)$$
$$=(f\bar{q}, \bar{Z}\bar{g})-\int\bar{q}dm\cdot(f, \bar{Z}\bar{g})=(f\bar{q}, \bar{Z}\bar{g}) \quad (\forall g\in H^2).$$

Thus $f\bar{q}\perp\bar{Z}\bar{H}^2=\bar{H}^2_0$. But $f\bar{q}\perp H^2$ as $f\perp qH^2$. Hence $f\bar{q}\perp H^2\oplus\bar{H}^2_0=L^2$. Therefore $f\bar{q}=0$ a.e., hence $f=0$ a.e.. Thus $H^2\ominus M=q\cdot H^2$.

Corollary. *The following properties are equivalent.*

(I) *H^2 and the classical Hardy space $H^2(d\theta)$ are isometrically isomorphic to each other.*

(II) *For every non trivial closed subspace N of H^2 invariant under multiplication by functions in A, $M=H^2\ominus N$ has the form*

$$M=P[Tq\cdot\bar{H}^2]$$

where q is an inner function.

Further, if these conditions hold, then every complementary in-

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*) (,) denotes the usual inner product in L^2.

variant subspace M is the closed linear span of $\{T^n q\}_{n=1}^{\infty}$ for some inner function q.

Proof. (I)$\Rightarrow$(II). It is easy to see that the simple invariance and the A-invariance are equivalent in the classical case. The assertion follows from Theorem.

(II)$\Rightarrow$(I). Suppose that (I) fails. Then $N=\left\{f\in H^2;\int f\cdot\bar{Z}^n dm=0(\forall n)\right\}$ is non trivial and A-invariant. By the assumption, $H^2\ominus N=P[Tq\cdot\bar{H}^2]$ for some inner function q. As in the proof of Theorem, we have $N=qH^2$. But this contradicts the fact that N is not simply invariant.

Now suppose that (I) or (II) holds. Then H^2 is the closed linear span of $\{Z^n\}_{n=0}^{\infty}$. It follows that M is the closed linear span of $\{P(Tq\cdot\bar{Z}^n)\}_{n=0}^{\infty}$. It suffices to see that for $n=0,1,2,\cdots$

$$P(Tq\cdot\bar{Z}^n)=T^{n+1}q. \tag{1}$$

Clearly $P(Tq)=Tq$. By the induction on n, we show that for $n=1,2,\cdots$,

$$Tq\bar{Z}^n=T^{n+1}q\oplus\left\{\sum_{j=1}^{n}\int T^j qdm\,\bar{Z}^{n-j+1}\right\}. \tag{2}$$

We have

$$Tq\cdot\bar{Z}=\frac{Tq-\int Tqdm}{Z}+\left(\int Tqdm\right)\cdot\bar{Z}$$

$$=T^2q\oplus\left(\int Tqdm\right)\cdot\bar{Z}.$$

Suppose $n>1$ and we know (2) for $n-1$. Then

$$Tq\cdot\bar{Z}^n=\frac{Tq\bar{Z}^{n-1}}{Z}=\frac{1}{Z}\left[T^n q+\left\{\sum_{j=1}^{n-1}\int T^j qdm\,\bar{Z}^{n-j}\right\}\right]$$

$$=\frac{T^n q-\int T^n qdm}{Z}+\bar{Z}\left[\int T^n qdm+\left\{\sum_{j=1}^{n-1}\int T^j qdm\,\bar{Z}^{n-j}\right\}\right]$$

$$=T^{n+1}q\oplus\left\{\sum_{j=1}^{n}\int T^j qdm\bar{Z}^{n-j+1}\right\}.$$

Thus (2) holds. This implies (1), completing the proof.

Remark. The first part of Collorary is suggested by Merrill [3] and the second part is the same of Theorem 4 in Douglas, Shapiro and Shields [2]. (See Ann. Inst. Fourier, **20**, 37–76 (1970) for the proof.)

References

[1] A. Browder: Introduction to Function Algebras. Benjamin, New York (1969).

[2] R. G. Douglas, H. S. Shapiro, and A. L. Shields: On cyclic vectors of the backward shift. Bull. Amer. Math. Soc., **73**, 156–159 (1967).

[3] S. Merrill: Maximality of Certain Algebras $H^\infty(dm)$. Math. Zeitschr., **106** 262–266 (1968).