178. *On the Adjoint Semigroups of Rings. I*

By Ferenc SZÁSZ
Budapest

(Comm. by Kinjirô KUNUGI, M. J. A., Sept. 12, 1970)

The fundamental notions, used in this paper, can be found in A. H. Clifford-G. B. Preston [1], N. Jacobson [3] and N. H. McCoy [4]. As it is well known, H. J. Hoehnke [2] has developed an interesting theory of a radical for semigroups with zero such that this radical is similar in certain sense to the Jacobson radical of rings (see [3]). Furthermore, H. Seidel [5] has introduced for semigroups, not necessarily having zero, the concept of right quasiregular elements, and he has proved [5] with the help of right quasiregular elements, that the Hoehnke radical of any semigroup S with zero coincides with the nil radical of S. (In addition see yet author's paper [7] on six further similar radicals of semigroups.) We recall, that an element $s\in S$ is right quasiregular in the semigroup S if and only if for arbitrary elements $t\in S$ and $u\in S$ nonnegative rational integers m and n there exist such that $s^m t=s^n u$ holds, eventually $s^0 t$ denoting the element t.

Some results on general radicals of semigroups with zero are discussed in author's paper [8].

A semigroup S having twosided unity element e is said to be *almost right quasiregular*, if $S=e\cup Q$ with $e\notin Q$ holds such that Q is a subsemigroup of S, and any element of Q is right quasiregular in S.

Furthermore, a semigroup S with both twosided unity element e and zero will be called almost nil (or almost nilpotent), if $S=e\cup N$ with $e\notin N$ holds, where N is a nil (or nilpotent, respectively) subsemigroup of S. Here N is said to have a bounded index m of nilpotency of elements, if a natural number m there exists such that $x^m=0\in N$ for any $x\in N$ holds.

Thirdly, a semigroup S with both twosided unity element e and zero will be called almost trivial, if $S=e\cup T$ with $e\notin T$ holds, where T is a subsemigroup of S such that all products xy for arbitrary $x\in T$ and $y\in T$ coincide with the zero element of S.

Therefore, any almost trivial semigroup is commutative.

All rings, considered here, will be associative. For any ring A, the elements of A form with respect to the circle operation $a\circ b=a+b-ab$ a semigroup S, which is called the adjoint semigroup of the ring A. Obviously the zero 0 of A is the twosided unity element of S. Furthermore an element e of A is a right zero of S if and only if e is a right unity element of A. A ring A will be called H-ring, if its adjoint semigroup S is almost right quasiregular. H-rings can be considered as very strongly nonradical rings for the Jacobson radical [3], the adjoint semigroup of any Jacobson radical ring being a group. For results on the circle operation we refer the reader for instance to [3], [6] and [9].

The aim of this paper is to point out, without proof, the equivalence of thirteen conditions for adjoint semigroups of rings. Namely we have the following:

Theorem. *The following thirteen conditions for an adjoint semigroup S of a ring $A(\neq 0)$ are mutually equivalent:*

(I) *S is almost right quasiregular.*

(II) *S is almost right quasiregular with an idempotent element, which differs from the unity element of S.*

(III) *S is almost right quasiregular with a left zero.*

(IV) *S is almost right quasiregular with a right zero.*

(V) *S is almost right quasiregular with zero.*

(VI) *S is almost nil.*

(VII) *S is almost nil with bounded index of nilpotency of elements.*

(VIII) *S is almost nilpotent.*

(IX) *S is almost nilpotent and commutative.*

(X) *S is almost trivial.*

(XI) *S is almost trivial and finite.*

(XII) *S is the adjoint semigroup of the field of two elements.*

(XIII) *S consists of two elements a and b with the multiplication:*

	a	b
a	a	a
b	a	b

Remark. We mention that the equivalence of conditions (VI) and (XII) is verified in author's paper [9].

References

[1] A. H. Clifford and G. B. Preston: The Algebraic Theory of Semigroups. I, II. Providence (1961), (1967).

[2] H. J. Hoehnke: Structure of semigroups. Canadian J. Math., **18**, 449–491 (1966).

[3] N. Jacobson: Structure of Rings, 2 edition. Providence (1964).

[4] N. H. McCoy: The Theory of Rings. Macmillan, New York (1964).

[5] H. Seidel: Über das Radikal einer Halbgruppe. Mathematische Nachrichten, **29**, 255–263 (1965).

[6] F. Szász: An observation on the Brown-McCoy radical. Proc. Japan Acad., **37**, 413–416 (1961).

[7] F. Szász: Radikalbegriffe für Halbgruppen mit Nullelement, die dem Jacobsonschen ringtheoretischen Radikal ähnlich sind. Math. Nachrichten, **34**, 157–161 (1967).
[8] ——: On radicals of semigroups with zero. I. Proc. Japan Acad., **46**, 595–598 (1970).
[9] ——: Rings with an almost nil adjoint semigroup. Proc. Japan Acad., **46**, 593–594 (1970).