175. *On a Theorem of Koebe for Quasiconformal Mappings*

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1. Let $w=f(z)$ be a quasiconformal mapping, whose dilatation quotient is bounded above by $K(\geqq 1)$, of the unit disc $|z|<1$ into a domain in $|w|<\infty$ in the sense of Grötzsch. We call such mapping a K-quasiconformal mapping of $|z|<1$ into $|w|<\infty$. If $w=f(z)$ is a K-quasiconformal mapping for some K, then it is called a quasiconformal mapping. We denote by S_α and $S_\alpha(K)$, respectively, the families of quasiconformal mappings and K-quasiconformal mappings in Grötzsch sense such that each mapping $f(z)$ is univalent in $|z|<1$ and $f(0)=0$ and $\lim_{z\to 0}|f(z)|/|z|^\alpha=1$, where α is real.

We denote by $\mathfrak{S}_\alpha$ and $\mathfrak{S}_\alpha(K)$, respectively, the families of quasiconformal mappings and K-quasiconformal ones of $|z|<1$ into a domain in $|w|<\infty$ in the sense of Pfluger-Ahlfors such that these mappings satisfy the same normalization as above at the origin. A univalent quasiconformal mapping in Grötzsch sense is a continuously differentiable quasiconformal one in Pfluger-Ahlfors sense. Then we have

$$S_\alpha\subset\mathfrak{S}_\alpha,\ S_\alpha(K)\subset\mathfrak{S}_\alpha(K)\quad\text{and}\quad \mathfrak{S}_\alpha(K)\subset\mathfrak{S}_\alpha.$$

2. Y. Juve [2] extended Koebe's quarter-disc theorem to the family $S_{1/d(0)}$, where $d(0)$ means the value at the origin of the dilatation quotient of $w=f(z)$, and proved the following theorem:

Theorem. *Let $w=f(z)$ be any mapping belonging to $S_{1/d(0)}$. Denote by δ the distance from the origin to the boundary of the image domain of $|z|<1$ under $w=f(z)$. Then*

$$\delta\geqq\frac{1}{4}\exp\left\{-\int_0^1\left(\frac{1}{d(0)}-\frac{1}{\frac{1}{2\pi}\int_0^{2\pi}d(z)d\theta}\right)\frac{dr}{r}\right\}.$$

Here, an extremal mapping giving the equality is the composite mapping of

$$\zeta=|z|^{1/d(0)}\{1+(R-1)|z|^{R/(1-R)d(0)}\}e^{i\arg z}$$

and $w=\zeta\Big/\left(1+\dfrac{\zeta}{R}\right)^2$, where

$$R=\exp\left\{-\int_0^1\left(\frac{1}{d(0)}-\frac{1}{\frac{1}{2\pi}\int_0^{2\pi}d(z)d\theta}\right)\frac{dr}{r}\right\}.$$

3. The proof for this theorem by Juve [2] is a modification of that for Ahlfors' distortion theorem (cf. R. Nevanlinna [3], Kap. IV,

§ 4. Verzerrungssätze von Ahlfors). It is well known that the dilatation quotient $d(z)$ of a quasiconformal mapping in Pfluger-Ahlfors sense can be defined almost everywhere. Whence, by proceeding similarly as Juve's proof, we can establish the following under the more general normalization:

Theorem 1. *Let $w=\varphi(z)$ be any mapping belonging to $\mathfrak{S}_\alpha$, and let δ denote the distance from the origin to the boundary of the image domain of $|z|<1$ under $w=\varphi(z)$. Then*

$$\delta \geqq \frac{1}{4} \exp\left(-\int_0^1 \left(\alpha - \frac{1}{\frac{1}{2\pi}\int_0^{2\pi} d(z)d\theta}\right)\frac{dr}{r}\right).$$

And an extremal mapping giving the equality is the composite mapping of

$$\zeta = |z|^\alpha \{1+(R-1)|z|^{\alpha R/(1-R)}\} e^{i\arg z}$$

and $w=\zeta\Big/\left(1+\dfrac{\zeta}{R}\right)^2$, where

$$R = \exp\left(-\int_0^1 \left(\alpha - \frac{1}{\frac{1}{2\pi}\int_0^{2\pi} d(z)d\theta}\right)\frac{dr}{r}\right).$$

Putting $\alpha=\dfrac{1}{d(0)}$ especially, Theorem 1 reduces to Juve's theorem.

Corollary. *Under the same notation and condition as in Theorem 1, if*

$$\alpha > \limsup_{r\to 0} \frac{1}{\frac{1}{2\pi}\int_0^{2\pi} d(z)d\theta},$$

then there is no so-called Koebe's constant, and if

$$\alpha < \liminf_{r\to 0} \frac{1}{\frac{1}{2\pi}\int_0^{2\pi} d(z)d\theta},$$

then $\delta=+\infty$.

4. Now, according to our paper [1], the family $\mathfrak{S}_\alpha(K)$ is empty if and only if $\alpha>K$ or $\alpha<\dfrac{1}{K}$. Then, Koebe's quarter-disc theorem can be extended to the family $\mathfrak{S}_\alpha(K)$ as follows:

Theorem 2. *The family $\mathfrak{S}_\alpha(K)$ has Koebe's constant if and only if $\alpha=\dfrac{1}{K}$, where Koebe's constant is equal to $\dfrac{1}{4}$ and an extremal mapping is the composite mapping of*

$$w=\frac{\zeta}{(1-\zeta/r^{1/K})^2} \quad \text{and} \quad \zeta=|z|^{1/K}e^{i\arg z}.$$

Proof. Since there holds clearly

$$\frac{1}{2\pi}\int_0^{2\pi} d(z)d\theta \leqq K,$$

it follows from Theorem 1 that

$$(*) \qquad \delta \geqq \frac{1}{4}\exp\left\{-\int_0^1\left(\alpha-\frac{1}{K}\right)\frac{dr}{r}\right\}.$$

Therefore we have $\delta \geqq \frac{1}{4}$ for $\alpha=\frac{1}{K}$.

For the case $\alpha>\frac{1}{K}$, the aboye estimate (*) asserts only $\delta \geqq 0$. But, it csn be concluded by mentioning the following concrete examples that $\delta=0$ in fact. In the case $\frac{1}{K}<\alpha\leqq 1$, consider

$$w_n(z)=|z|^\alpha\left\{1-\left(1-\frac{1}{n}\right)|z|^{(\alpha K-1)/K(n-1)}\right\}e^{i\alpha rgz}.$$

And in the case $1<\alpha\leqq K$, take

$$w_n(z)=\begin{cases} z\,|z|^{\alpha-1} & \text{for} \quad |z|<\dfrac{1}{n}, \\ z\left(\dfrac{1}{n}\right)^{\alpha-1} & \text{for} \quad \dfrac{1}{n}\leqq |z|<1. \end{cases}$$

Then it can be easily verified that in either case $w_n(z)\in\mathfrak{S}_\alpha(K)$ and $|w_n(re^{i\theta})|\to 0$ as $n\to\infty$.

Corollary (A precision of Pfluger's result [4]). For any mapping $\varphi(z)$ of $\mathfrak{S}_\alpha(K)$, there holds $\min\limits_{0<|z|=r<1}|\varphi(z)|\geqq\frac{1}{4}\left\{\frac{4r}{(1+r)^2}\right\}^{1/K}$ if and only if $\alpha=\frac{1}{K}$, where this estimate is sharp and there is no any positive lower bound of $|\varphi(z)|$ for $\alpha>\frac{1}{K}$.

References

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[3] Nevanlinna, R.: Eindeutige analytische Funktionen, Springer, 1 Aufl. (1936), 2 Aufl. (1953).

[4] Pfluger, A.: Quasikonforme Abbildungen und logarithmische Kapazität. Ann. Inst. Fourier, **2** (1951).