206. Note on Commutative Regular Ring Extensions of Rings

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Let R be a commutative ring with identity, and let S be a commutative ring extension of R with the same identity. If the canonical injection $R \to S$ is a flat epimorphism (in the category of commutative rings with identity), then, by Lazard [3], for any ideal B of S

$$B \to B \cap R \qquad (*)$$

is an injective mapping from the set of ideals of S into that of R.

In case S is a regular (in the sence of von Neumann) ring extension of R, we can give certain conditions that are necessary and sufficient for the mapping $(*)$ to be injective.

It is easily seen that if S is the classical ring of quotients of R the mapping $(*)$ is injective. There is an example of a commutative ring extension of R for which the mapping $(*)$ is a bijection but which does not coincide with the classical ring of quotients of R.

Throughout this note, a ring considered will mean a commutative ring with identity and its ring extension will mean a commutative one with the same identity.

1. Let R be a ring. For its ring extension S, we shall consider the following conditions:

(α) $B=(B \cap R)S$ *for any ideal B of S.*

(β) $A=AS \cap R$ *for any ideal A of R.*

The conditions (α) and (β) that the mapping $(*)$ are injective and surjective, respectively.

For a regular ring extension S of R, the spectrum X of S, i.e. the space of prime (=maximal) ideals with the hull-kernel topology, is compact, Hausdorff and extremely disconnected and S may be identified with the ring of global sections of a sheaf of fields over X (see [6] for a detailed discussion).

For $s \in S$ and $x \in X$, let s_x be the image of s under the natural homomorphism of S onto S/x, and let $S(s)$ be the support of s, i.e.

$$S(s)=\{x \in X \mid s_x \neq 0\}=\{x \in X \mid s \notin x\}.$$

Now we shall quote Mewborn [5, Theorem 3.1] as follows.

Lemma 1.1. *Let R be a ring and let S be a regular ring extension of R. Then S is flat as an R-module if and only if for any $a \in R$, there exists a finitely generated ideal I in R such that*

$$X-S(a)=S(I)(=\cup\{S(r)\mid r\in I\}).$$

Theorem 1.2. *Let R be a ring and let S be a regular ring extension of R. Then the following conditions are equivalent:*

(1) *S satisfies the condition (α)*

(2) *S is flat as an R-module and, for any idempotent e in S, we have $e\otimes 1=1\otimes e$ in $S\otimes_R S$.*

(3) *For any idempotent e in S, there exists a finitely generated ideal I in R such that $S(e)=S(I)$.*

Proof. (1)$\rightleftarrows$(3) Since S is regular,

$B=(B\cap R)S$ for any ideal B in S,

$\rightleftarrows bS=(bS\cap R)S$ for any element b in S,

$\rightleftarrows eS=(eS\cap R)S$ for any idempotent e in S,

$\rightleftarrows$for any idempotent e in S, there exists a finitely generated ideal I in R such that $eS=IS$,

$\rightleftarrows$for any idempotent e in S, there exists a finitely generated ideal I in R such that $S(e)=S(I)$,

by [6; Proposition 9.3].

(2)$\Rightarrow$(3) For any idempotent e in S, we have

$$(1-e)\otimes e=0.$$

Since S is flat as an R-module, there exist $a_1, a_2, \cdots, a_n$ in R and $s_1, s_2, \cdots, s_n$ in S such that

$$e=a_1s_1+a_2s_2+\cdots+a_ns_n,$$
$$a_i(1-e)=0\ (i=1,2,\cdots,n)$$

by [2; Chap. 1, §2, Proposition 13]. Since $a_i(1-e)=0$ means that $a_i\in eS\cap R$, we have $eS\subseteqq(eS\cap R)S$ and hence we have $eS=(eS\cap R)S$.

(3)$\Rightarrow$(2) First we shall show that S is flat as an R-module. By Lemma 1.1, we may show that, for any element a in R, there exists a finitely generated ideal I in R such that $S(a)=X-S(I)$.

In case $X=S(a)$, we may put $I=(0)$. Let us suppose that $X\neq S(a)$ and let x be in $X-S(a)$. Then $a_x=0$ and there exists some idempotent $e(x)$ in x such that $a(1-e(x))=0$. Thus we have

$$X-S(a)=\bigcup_{x\in X-S(a)}S(1-e(x)).$$

Since $S(a)$ is an open and closed set, $X-S(a)$ is compact, and hence there exists a finite number of idempotents $e_1, e_2, \cdots, e_n$ in S such that

$$X-S(a)=\bigcup_{i=1}^{n}S(1-e_i).$$

We have, by the assumption, $S(e_i)=S(I_i)$ for some finitely generated ideal I_i in $R(i=1,2,\cdots,n)$. Then $I=\sum_{i=1}^{n}I_i$ is also finitely generated and we have

$$X-S(a)=S(I).$$

Next, we shall show that $e\otimes 1=1\otimes e$ for any idempotent e in S. Let e be an idempotent in S, then by the assumption we have eS

$=(eS\cap R)S$, and hence there exist $a_1, a_2, \cdots, a_n$ in $eS\cap R$ and $s_1, s_2, \cdots, s_n$ in S such that

$$e=a_1s_1+a_2s_2+\cdots+a_ns_n.$$

Then

$$e\otimes(1-e)=(a_1s_1+a_2s_2+\cdots+a_ns_n)\otimes(1-e)$$
$$=s_1\otimes a_1(1-e)+\cdots+s_n\otimes a_n(1-e)=0.$$

This implies that $e\otimes 1=e\otimes e$. Similarly $1\otimes e=e\otimes e$, and hence we have $e\otimes 1=1\otimes e$.

Remark. Lazard [3] has shown that if the canonical injection $R\to S$ is a flat epimorphism, then S satisfies the condition (α). But the converse does not hold in general. For example, let R be a field and S a proper extension field of R. Then S satisfies the condition (α) but $R\to S$ is not a flat epimorphism.

2. In this section we shall consider the case in which R is also regular.

Proposition 2.1. *Let R be a regular ring and S a ring extension of R. Then S satisfies the condition (β).*

Proof. Let A be an ideal of R and let a be in $AS\cap R$. Then there exist $a_1, a_2, \cdots, a_n$ in A and $s_1, s_2, \cdots, s_n$ in S such that

$$a=a_1s_1+a_2s_2+\cdots+a_ns_n.$$

Since R is regular, we can find an idempotent e in A such that

$$a_ie=a_i \ (i=1, 2, \cdots, n).$$

Thus we have $a=ae$, and hence $a\in A$.

Proposition 2.2. *Let R be a regular ring and S a regular ring extension of R. Then each of the conditions* (1), (2), *and* (3) *in Theorem* 1.2 *is also equivalent to each of the following*:

(4) *Every idempotent in S is contained in R.*

(5) *For any idempotent e in S, we have $(R:e)S=S$, where $(R:e)$ is the ideal of R consisting of all the elements a in R such that $ae\in R$.*

Proof. (2)$\Rightarrow$(4) Let e be any idempotent in S. As is in the proof of Theorem 1.2, there exist $a_1, a_2, \cdots a_n$ in R and $s_1, s_2, \cdots s_n$ in S such that

$$e=a_1s_1+a_2s_2+\cdots+a_ns_n,$$
$$a_i(1-e)=0 \ (i=1, 2, \cdots, n).$$

It follows that

$$a_i\in(R:e) \quad (i=1, 2, \cdots, n).$$

Since R is regular, we can find an idempotent f in $(R:e)$ such that

$$a_if=a_i \quad (i=1, 2, \cdots, n).$$

Thus we have $e=ef\in R$.

(4)$\Rightarrow$(5) Obvious.

(5)$\Rightarrow$(2) Suppose that $(R:e)S=S$ for any idempotent e in S. Then there exist $a_1, a_2, \cdots, a_n$ in $(R:e)$ and $s_1, s_2, \cdots, s_n$ in S such that

$$1 = a_1 s_1 + a_2 s_2 + \cdots + a_n s_n.$$

Thus we have

$$\begin{aligned} e \otimes 1 &= e \otimes (a_1 s_1 + a_2 s_2 + \cdots + a_n s_n) \\ &= e a_1 \otimes s_1 + \cdots + e a_n \otimes s_n \\ &= 1 \otimes e a_1 s_1 + \cdots + 1 \otimes e a_n s_n \\ &= 1 \otimes e. \end{aligned}$$

Corollary 2.3. *Let R be a Boolean ring and let S be a ring extension of R which is essential over R as an R-module. Then S satisfies the condition (α) if and only if $R=S$.*

Proof. Suppose that S satisfies the condition (α). Since S is essential over R, S is contained in the maximal ring of quotients $Q(R)$ of R (see [4], p. 99), and S is also a Boolean ring (see [4], p. 44). Thus we have, by Proposition 2.2, $S=R$.

Finally, we shall make mention of an example of a ring extension of R for which the mapping ($*$) is a bijection but which does not coincide with the classical ring of quotients of R.

Example. Let R be a continuous regular ring which is not self-injective (see [7], Example 3). Since R is regular, $Q(R)$ is self-injective and the classical ring of quotients $C(R)$ coincides with R. Hence we have $Q(R) \neq C(R)$. By [7; Lemma 8], every idempotent in $Q(R)$ is contained in R, and by Propositions 2.1 and 2.2, $Q(R)$ satisfies the conditions (α) and (β).

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