205. *On Potent Rings. III*

By Hidetoshi MARUBAYASHI
College of General Education, Osaka University

(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1970)

In [5], [6], we have mainly investigated potent irreducible rings. The purpose of this paper is to prove that a right locally uniform potent ring with zero right singular ideal is an essential irredundant subdirect sum PI-rings and conversely. A number of concepts and results are needed from [5] and [6].

By the same argument as in Theorem 2.2 of [2], we obtain the following

Proposition 1. *Let R be a right locally uniform ring with $Z_r(R)=0$, let I be a right ideal of R and let I^* be a unique maximal essential extension of I in R. Then $I^*=\{a\in R\mid aE\subseteqq I$ for some $E\subset' R\}$.*

Let R be a right locally uniform ring with $Z_r(R)=0$ and let $\hat{R}$ be the maximal right quotient ring of R. Then the mappings

$$A\to E_R(A),\ A\in L_r^*(R)\,;\ \hat{A}\to\hat{A}\cap R,\ \hat{A}\in L_r^*(\hat{R})$$

are mutually inverse isomorphisms between $L_r^*(R)$ and $L_r^*(\hat{R})$, where $E_R(A)$ is a right R-injective hull of A (see [1]). Let A be an element of $L_r^*(R)$. Then we denote by $\hat{A}$ the element of $L_r^*(\hat{R})$ which corresponds to A. Clearly $\hat{A}$ is a right R-injective hull of A and is right $\hat{R}$-injective. Let A and B be uniform right ideals of R. As in [5], A and B are similar (in symbol; $A\sim B$) iff A and B contain mutually isomorphic nonzero right ideals A' and B', respectively. The set of all uniform right ideals of R can be classified by the equivalence relation $\sim$. $\{A_i\}$ will denote the class containing the uniform right ideal A_i. We now set $R_i=(\sum_{A\in\{A_i\}}A)^*$. Then we obtain

Proposition 2. *Let R be a right locally uniform ring with $Z_r(R)=0$. Then the following properties hold:*

(1) *$\sum_{A\in\{A_i\}}A$ is a two-sided ideal.*

(2) *R_i is an ideal of R for each i.*

(3) *If B is a uniform right ideal of R and if $B\subseteqq R_i$, then $B\sim A_i$.*

(4) *$\sum_i R_i$ is a direct sum.*

Proof. Let A be a uniform right ideal and let x be an element of R. Then $xA=0$ or $xA\cong A$ and hence (1) follows immediately.

(2) follows immediately from Proposition 1 and (1).

(3) is obtained by the same argument as in Lemma 5.5 of [3].

(4) We can prove that $\hat{R}_i$ is an $\hat{R}$-injective hull of the sum of all minimal right ideals of $\hat{R}$ which are isomorphic to $\hat{A}_i$. Hence the

sum of $\hat{R}_i$ is a direct sum and therefore $\sum_i R_i$ is a direct sum.

Proposition 3. *If R is a right locally uniform ring with $Z_r(R)=0$, then the followings hold:*

(1) *$\hat{R}_i$ is right self-injective, regular and prime as a ring.*

(2) *$\hat{R}_i$ is the maximal right quotient ring of R_i for each i.*

(3) *$L_r^*(R_i)=\{I \in L_r^*(R) \mid I \subseteq R_i\}$.*

(4) *If R is a potent ring, then R_i is a PI-ring.*

Proof. (1) Since $\hat{R}_i$ is an $\hat{R}$-injective hull of the sum of all minimal right ideals which are isomorphic to $\hat{A}_i$, $\hat{R}_i$ is an ideal of $\hat{R}$ and is a direct summand of $\hat{R}$. From these (1) follows immediately.

(2) Since $\hat{R}_i$ is a regular ring and is a right self-injective ring by (1), it is enough to prove that $\hat{R}_i' \supset R_i$ as right R_i-modules. Let q be a nonzero element of $\hat{R}_i$. Then there exists $r \in R$ such that $0 \neq qr \in R \cap \hat{R}_i = R_i$. Since $R_iR_j=0$ $(i \neq j)$, $\sum_i R_i \subset' R$ and $Z_r(R)=0$, we obtain $qrR_i \neq 0$. Hence there exists $r' \in R_i$ such that $0 \neq (qr)r' = q(rr') \in R_i$ and $rr' \in R_i$, as desired.

(3) Let I be a closed right ideal of R such that $I \subseteq R_i$. Then $\hat{I}$ is a direct summand of $\hat{R}_i$ and $I=\hat{I} \cap R=(\hat{I} \cap \hat{R}_i) \cap R=\hat{I} \cap (\hat{R}_i \cap R)=\hat{I} \cap R_i$. Hence we have $I \in L_r^*(R_i)$. Conversely, let I be a closed right ideal of R_i and let $\bar{I}=E_{R_i}(I)$. Then $\bar{I}$ is a right ideal of $\hat{R}$ and is a direct summand of $\hat{R}$. Since $\bar{I} \cap R=(\bar{I} \cap \hat{R}_i) \cap R=\bar{I} \cap (\hat{R}_i \cap R)=\bar{I} \cap R_i=I$, we obtain $I \in L_r^*(R)$ and $I \subseteq R_i$, as desired.

(4) follows from (1) and (3).

We shall call R_i an irreducible component of R.

Let R be a right locally uniform potent ring with $Z_r(R)=0$. Then R is said to be locally residue-finite iff the irreducible components R_i of R are residue-finite as a ring. By Proposition 3, if R is a right locally uniform potent ring with $Z_r(R)=0$ and if R is locally residue-finite, then R_i is a residue-finite PI-ring for each i. Now we set $P_i=(\sum_{j \neq i} R_j)^*$ and $\bar{R}_i=R/P_i$ for each i. Then the followings hold:

(i) $\bigcap_i P_i=0$.

(ii) $\bigcap_{j \neq i} P_j \neq 0$.

(iii) $\bar{R}_i' \supset R_i$ as a right R_i-module for each i.

(iv) If R_i is a residue-finite PI-ring, then so is $\bar{R}_i$.

Let R be a subdirect sum of a family R_i of rings (that is $R \subset \prod_i R_i$ and the projection $R \to R_i$ is onto for each i). The subdirect sum will be called essential irredundant iff $\prod_i R_i' \supset \sum_i \oplus (R \cap R_i)$ as a right R-module (see [1]).

Now, we can summarize the above-mentioned results as follows:

Theorem 1. *Let R be a right locally uniform potent ring with $Z_r(R)=0$ and $\{\bar{R}_i\}$ be as above. Then R is an essential irredundant subdirect sum of $\{\bar{R}_i\}$, where $\bar{R}_i$ is a PI-ring for each i. Furthermore if R is locally residue-finite, then $\bar{R}_i$ is a residue-finite PI-ring.*

We now give a converse of Theorem 1.

Theorem 2. *Let $\{\bar{R}_i\}$ be a family of PI-rings and let R be an essential irredundant subdirect sum of $\{\bar{R}_i\}$. Then*

(1) *R is a right locally uniform potent ring with $Z_r(R)=0$.*

(2) *If $\bar{R}_i$ is residue-finite for each i, then R is locally residue-finite.*

Proof. We first prove that R is a right locally uniform ring with $Z_r(R)=0$. Let $\hat{\bar{R}}_i$ be the maximal right quotient ring of $\bar{R}_i$ for each i. Then $\hat{\bar{R}}_i$ is a full left linear ring over a division ring. We set $S=\prod_i \bar{R}_i$. Then, by ([4; p. 72, Proposition]), $\hat{S}=\prod_i \hat{\bar{R}}_i$ is the maximal right quotient ring of S. By ([1, p. 117, Theorem 3.9]), $\hat{S}$ is right self-injective, right locally uniform and regular as a ring. Since, by the assumption, $S\,'\supset \sum_i \oplus(\bar{R}_i \cap R)$ as a right R-module, $\hat{S}$ is the maximal right quotient ring of R and hence R is a right locally uniform ring with $Z_r(R)=0$. Let I be a closed right ideal of R and let $I_i=\{x_i \in \bar{R}_i \,|\, a=(x_i)\in I$ for some $a \in I\}$. Then we can prove that I_i is a closed right ideal of $\bar{R}_i$. Hence R is a potent ring. Now we set $R_i=\bar{R}_i\cap R$. Then the following properties hold.

(1) $R_i \in L_{r2}^*(R)$ and $\bar{R}_i$ is a right quotient ring of R_i for each i.

(2) $\{R_i\}$ are the irreducible components of R.

Hence, by Proposition 3, we obtain $L_r^*(R_i)=\{I\in L_r^*(R)\,|\, I\subseteq R_i\}$. Furthermore, we can prove that $\bar{T}=\hat{T}\cap \bar{R}_i$ is a closed ideal of R_i for each $T\in L_{r2}^*(R_i)$. Since $L_r^*(R_i)\cong L_r^*(\bar{R}_i)$, R_i is residue-finite if $\bar{R}_i$ is residue-finite. Hence if $\bar{R}_i$ is residue-finite for each i, then R is locally residue-finite.

References

[1] C. Faith: Lectures on Injective Modules and Quotient Rings. Springer-Verlag, New York (1967).

[2] A. W. Goldie: Semi-prime rings with maximum condition. Proc. Lond. Math. Soc., **10**, 201–220 (1960).

[3] ——: Torsion-free modules and rings. J. Algebra, **1**, 268–287 (1964).

[4] L. Levy: Unique direct sums of prime rings. Trans. Amer. Math Soc., **106**, 64–76 (1963).

[5] H. Marubayashi: On potent rings. I. Proc. Japan Acad., **46**, 893–896 (1970).

[6] ——: On potent rings. II. Proc. Japan Acad., **46** 897–900 (1970).