204. On Potent Rings. II

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In [7], we defined residue-finite *CPI*-rings which are s-complemented with respect to L^*_{r2}. In this paper we shall give characterizations of such rings. Let R be a residue-finite *CPI*-ring and let $\hat{R}$ be the maximal right quotient ring of R. We shall give also a necessary and sufficient condition that $\hat{R}$ is a left quotient ring of R. This is a generalization of Faith's result [1] on prime rings. Terminology and notation will be taken from [6] and [7].

1. Triangular-block matrix rings with infinite dimension.

We shall give examples of residue-finite *CPI*-rings which are s-complemented with respect to L^*_{r2}. Let F be a division ring and let ω be a countable ordinal number. We denote by $(F)_\omega$ the ring of all column-finite $\omega\times\omega$ matrices over F. Let F_{ij} be additive subgroups of F such that

(1.1) $\qquad F_{ij}F_{jk}\subseteqq F_{ik} \qquad (i,j,k=1,2,\cdots).$

Let

(1.2) $\qquad S=\{a\in(F)_\omega \mid a=(a_{ij}),\ a_{ij}\in F_{ij}\}.$

Clearly S is the subring of $(F)_\omega$. The ring S will be called a T-ring (triangular-block matrix ring) with type (A) in $(F)_\omega$ iff there exist integers $0=d_0<d_1<\cdots<d_n<\cdots$ such that

(1.3) $F_{ij}\neq 0$ iff $i>d_p$ and $d_p<j\leqq d_{p+1}$ $(p=0,1,2,\cdots)$.

The ring S will be called a T-ring with type (B) in $(F)_\omega$ iff there exist integers $0=d_0<d_1<\cdots<d_p$ such that

(1.4) $F_{ij}\neq 0\Longleftrightarrow$(i) if $j\leqq d_p$, then $i>d_k$ and $d_{k-1}<j\leqq d_k$ for some $k(1\leqq k\leqq p)$, (ii) if $j>d_p$, then $i>d_p$.

In both cases, associated with S is the full T-ring

(1.5) $M=\{a\in(F)_\omega \mid a=(a_{ij}),\ a_{ij}\in F'_{ij}\}$, where $F'_{ij}=F$ whenever $F_{ij}\neq 0$ and $F'_{ij}=0$ otherwise.

Following R. E. Johnson, we shall call M the full cover of S. Let A and B be subsets of a division ring F. The set $\{ab^{-1}\mid a\in A, 0\neq b\in B\}$ will be denoted by AB^{-1}. A ring Q is called a right quotient ring of a subring R if for each $a, 0\neq b\in Q$, there exist $r\in R$ and $n\in Z$ such that $ar+na\in R$ and $br+nb\neq 0$, where Z is the ring of integers; in symbols: $R\leqq Q$. A left quotient ring is defined similarly. If Q is a left and right quotient ring of a ring R, then we write $R\leqq_l Q$. If R has the zero right singular ideal, then Q is a right quotient ring of R if and only if Q is a right quotient ring of R in the sense of R. E. Johnson (see [1]). It is well known that an infinite-dimensional I-ring R has a full ring $\hat{R}$ of linear transformations of an infinite-dimensional vector space over a division ring as a maximal right quotient ring and that $L_r^*(\hat{R}) \cong L_r^*(R)$ under the correspondence $\hat{A} \to \hat{A} \cap R$, $\hat{A} \in L_r^*(\hat{R})$. Let A be an element of $L_r^*(R)$. Then we denote by $\hat{A}$ an element of $L_r^*(\hat{R})$ which corresponds to A. As is well known, $\hat{A}$ is a right R-injective hull of A and is right $\hat{R}$-injective. Since $(F)_\omega$ is a column-finite, we obtain the following four theorems by the same arguments as in Theorems 3.5, 3.7 and 3.9 of [6].

Theorem 1. *Let S be a T-ring $(F)_\omega$ given by* (1.3) *or* (1.4). *Then $S \leqq (F)_\omega$ if and only if $F_{11}F_{11}^{-1} = F$.*

Theorem 2. *Let S be a T-ring in $(F)_\omega$ given by* (1.3) *or* (1.4) *such that $S \leqq (F)_\omega$. Then S is potent if and only if $F_{jj}F_{kj}^{-1} = F$ for $j < k$ $(j, k = 2, 3, \cdots)$.*

Theorem 3-1. *Let S be a T-ring with type* (A) *in $(F)_\omega$ whose blocks are defined by the numbers $0 = d_0 < d_1 < \cdots < d_n < \cdots$ in* (1.3). *If $S \leqq (F)_\omega$ and if S is potent, then $L_{r2}^* = \{T_0, T_1, T_2, \cdots, T_n, \cdots\}$, where $T_0 = R$ and $T_p = \{a \in S \mid a = (a_{ij}), a_{ij} = 0 \text{ if } i \leqq d_p\}$ for every p.*

Theorem 3-2. *Let S be a T-ring with type* (B) *in $(F)_\omega$ whose blocks are defined by the numbers $0 = d_0 < d_1 < \cdots < d_p$ as in* (1.4). *If $S \leqq (F)_\omega$ and if S is potent, then $L_{r2}^* = \{T_0, T_1, T_2, \cdots, T_p, T_{p+1}\}$, where $T_0 = R$, $T_{p+1} = 0$ and $T_k = \{a \in S \mid a = (a_{ij}), a_{ij} = 0 \text{ if } i \leqq d_k\}$ for $1 \leqq k \leqq p$.*

2. Residue-finite CPI-rings as matrix rings.

Let R be a residue-finite CPI-ring which is s-complemented with respect to L_{r2}^* and let $\{A_i\}$ and $\{B_i\}$ be as given in Theorem 2 of [7]. Then $\{\hat{A}_1, \hat{A}_2, \cdots, \hat{A}_n, \cdots\}$ is an atomic basis of $L_r^*(\hat{R})$ which corresponds to the atomic basis $\{A_1, A_2, \cdots, A_n, \cdots\}$ of $L_r^*(R)$. By Theorem 1, 11 of [1; p. 108], there exist matrix units $\{e_{ij} \mid i, j = 1, 2, \cdots\}$ in $\hat{R}$ such that $\hat{A}_i = e_{ii}\hat{R}$ and $\hat{R} = (F)_\omega$, where F is a division ring. Clearly $A_i = e_{ii}\hat{R} \cap R$ and $B_i = (\bigcup_{j \neq i} A_j)^l = \hat{R}e_{ii} \cap R$. Let $A_i \cap B_j = F_{ij}e_{ij}$ $(i, j = 1, 2, \cdots)$. Then F_{ij} are additive subgroups of F satisfying (1.1). If we put $S = \{a \in R \mid a = (a_{ij}), a_{ij} \in F_{ij}\}$, then S is a subring of R. By Theorem 2 of [7], $F_{ij} \neq 0$ if and only if $i > d_0 + d_1 + \cdots + d_p$ and $d_0 + d_1 + \cdots + d_p < j \leqq d_0 + d_1 + \cdots + d_{p+1}$ for some p. Thus, S is a T-ring in $(F)_\omega$ with the same block numbers as in R. Let M be the full cover of S. Then we have $S \leqq R \leqq M \leqq \hat{R}$.

Theorem 4. *Let R be a left faithful ring and let $\hat{R}$ be the maximal right quotient ring of R. Then in order that R is a residue-finite CPI-ring with type* (A) *and s-complemented with respect to L_{r2}^*, it is necessary and sufficient that $S \leqq R \leqq M \leqq \hat{R} = (F)_\omega$, where F is a division ring, S is a potent T-ring with type* (A) *in $(F)_\omega$ and M is the full cover of S.*

Lemma 1. *Let R be a residue-finite CPI-ring with type (A). If R is complemented with respect to L_{r2}^* and if $\hat{R}$ is a left self-injective ring, then R is s-complemented with respect to L_{r2}^*.*

By Theorem 4 and Lemma 1, we have;

Theorem 5. *Let R be a left faithful ring and let $\hat{R}$ be the maximal right quotient ring of R. If $\hat{R}$ is a left self-injective ring, then R is a residue-finite CPI-ring with type (A) and is complemented with respect to L_{r2}^* if and only if $S \leqq R \leqq M \leqq \hat{R} = (F)_\omega$, where F is a division ring, S is a potent T-ring with type (A) in $(F)_\omega$ and M is the full cover of S.*

Theorem 6. *Let R be a residue-finite CPI-ring and let $\hat{R}$ be the maximal right quotient ring of R. If $\hat{R}$ is a left quotient ring of R, then R is of type (B).*

By Theorem 6, we have;

Theorem 7. *Let R be a left faithful ring and let $\hat{R}$ be the maximal right quotient ring of R. Then R is a residue-finite CPI-ring and $\hat{R}$ is a left quotient ring ofR if and only if $S \leqq_l R \leqq_l M \leqq_l \hat{R} = (F)_\omega$, where F is a division ring, S is a potent T-ring with type (B) in $(F)_\omega$ and M is the full cover of S.*

3. Left and right quotient rings.

In view of Theorem 7, it is interesting to consider conditions which imply that the maximal right quotient ring is also a left quotient ring.

Theorem 8. *Let R be a residue-finite CPI-ring and let $\hat{R}$ be the maximal right quotient ring of R. Then $\hat{R}$ is a left quotient ring of R if and only if the following two conditions are satisfied.*

(1) *There exists an atom A of L_r^* such that $A^r = 0$.*

(2) *Let A be an atom satisfying $A^r = 0$. Put $\Gamma = \mathrm{Hom}_R(A, A)$ and $\Delta = \mathrm{Hom}_{\hat{R}}(\hat{A}, \hat{A})$. Then Δ is a left quotient ring of Γ and $\Delta A = \hat{A}$.*

Proof. Assume that $\hat{R}$ is also a left quotient ring of R. Let $\{A_i\}$ and $\{B_i\}$ be potent atom-bases of L_r^* and J_l^* respectively as in the proof of Theorem 7. For each nonzero $x \in A_i \cap B_i$, $x^l = (e_{ii}\hat{R})^l = \hat{R}(1-e_{ii}) \cap R$ is a maximal closed left ideal of R. Since $Z_l(R) = 0$ by Lemma 1.2 of [5], R^1x is a uniform left ideal of R by Theorem 6.9 of [3], where R^1x is the principal left ideal generated by x. Therefore, since $(\sum_{i=1}^\infty A_i \cap B_1)^l = 0$, R is a left stable ring in the sense of R. E. Johnson [4]. Now, By Theorem 6, R is of type (B) and hence there exists an atom A of L_r^* such that $A^r = 0$. Let θ and ϕ be nonzero elements of Γ and let u be a nonzero element of A. Then $\theta(u) \neq 0$, $\phi(u) \neq 0$, because every nonzero element of Γ is a non-singular mapping by Lemma 5.4 of [2]. Since $(\theta u)^r = u^r$, we obtain $(\theta u)^r = (\phi u)^r$ and $(\theta u)^{rl} = (\phi u)^{rl}$. Since u^r is a maximal closed right ideal, $(\theta u)^{rl}$ is a minimal annihilator left ideal and hence $(\theta u)^{rl} = (\phi u)^{rl}$ is an atom of L_l^* by Corollary 2.3 of [4]. Hence there exist $a, b \in R$ such that $a\theta(u) = b\phi(u) \neq 0$. Since $A^r = 0$, $Aa\theta(u) \neq 0$, and hence there exists $v \in A$ such that $va\theta(u)=vb\phi(u)\neq 0$. This means that $(\lambda_{va}\theta)(u)=(\lambda_{vb}\phi)(u)$, where $\lambda_{va}(x)=vax$ for $x \in A$. From which we obtain $\lambda_{va}\theta=\lambda_{vb}\phi$, because the elements of Γ, other than zero, are non-singular mappings. Evidently $\lambda_{va}, \lambda_{vb} \in \Gamma$ and $\Gamma\theta \cap \Gamma\phi \neq 0$; thus Γ is a left Ore domain. Let δ be any nonzero element of Δ. Since $\hat{A}$ is $\hat{R}$-right injective, there exists $e=e^2 \in \hat{R}$ sucn that $\hat{A}=e\hat{R}$. For $0\neq\delta(e)$, there exists $r \in R$ such that $0\neq r\delta(e) \in R$. Since $A^r=0$, there exists $a \in A$ such that $0\neq ar\delta(e) \in A$ and $0 \neq ar \in A$. Clearly $\lambda_{ar}\delta \in \Gamma$, $\lambda_{ar} \in \Gamma$ and $\lambda_{ar}\neq 0$, because $0\neq\lambda_{ar}\delta(e)$. This means that Δ is a left quotient ring of Γ. Evidently $\Delta A \subseteq \hat{A}$. Assume that q is a nonzero element of $\hat{A}$. Then there exists $r \in R$ such that $0\neq rq \in R$. Since $A^r=0$, $Arq\neq 0$ and there exists $u \in A$ such that $0\neq urq$. Since q^r is a maximal closed right ideal, $(urq)^r=(rq)^r=q^r$. Now define $\phi: urq\hat{R}\to\hat{A}$ by $\phi(urqy)=qy$ for each $y \in \hat{R}$. Then since $\hat{A}$ is right $\hat{R}$-injective, ϕ can be extended to $\hat{\phi} \in \Delta$ and $\hat{\phi}(urq)=\phi(urq)=q$, $urq \in A$. This means that $\Delta A \supseteq \hat{A}$. Hence we have $\Delta A=\hat{A}$, as desired.

Conversely, assume that (1) and (2) hold. If $0\neq q \in \hat{R}$, then since $A^r=0$, we have $Aq\neq 0$. There exists $a \in A$ such that $w=aq\neq 0$. Since $w \in \hat{A}=\Delta A$, there exist $\delta_1, \cdots, \delta_n \in \Delta$ and $a_1, \cdots, a_n \in A$ such that $w=\sum_{i=1}^{n}\delta_i a_i$. Now Δ is a left quotient ring of Γ. Hence there exists $0\neq\gamma \in \Gamma$ such that $0\neq\gamma\delta_i=\gamma_i \in \Gamma$, $i=1,\cdots,n$. Since $\Gamma A \subseteq A$, we obtain that $0\neq\gamma w=(\gamma a)q=\Sigma\gamma_i a_i \in Aq \cap A$. Thus we have $Rq \cap R\neq 0$. This means that $\hat{R}$ is a left quotient ring of R.

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