# 203. *On Potent Rings. I*[*)]

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(Comm. by Kenjiro SHODA, M. J. A., Oct. 12, 1970)

A ring $R$ is said to be (right) *potent* iff every nonzero closed right ideal $A$ of $R$ is potent, that is, $A^n$ is not zero for all positive integer $n$. In [6], R. E. Johnson has investigated potent irreducible rings which are finite dimensional in the sense of Goldie [4], and obtained many interesting results. The aim of this paper is to generalize the Johnson's work [6] to the case of the rings with infinite dimensions.

**1. Definitions and notations.**

Let $R$ be an associative ring. A right ideal $I$ of $R$ is called *closed* if it has no proper essential extensions in $R$ as right $R$-modules. Clearly the concept of closed right ideals of $R$ coincides with the one of complemented right ideals in the sense of Goldie [4]. A right ideal $E$ of $R$ is called *large* if $R$ is an essential extension of $E$ (in symbols; $E \subset' R$). A ring $R$ is said to be *(right) locally uniform* if any nonzero right ideal of $R$ contain a nonzero uniform right ideal. A right ideal $A$ is *uniform* if $A$ is an essential extension of every nonzero right ideal contained in $A$. Clearly, if $R$ is finite dimensional, then $R$ is locally uniform. $R$ is called *countably dimensional* if $R$ has a direct sum of countable right ideals. The notation $A^r(A^l)$ is used for right (left) annihilator of a subset $A$ of $R$.

The set $Z_r(R)=\{x \in R \mid x^r: \text{large right ideal of } R\}$ is an ideal of the ring $R$, which is called the *right singular ideal*. If $Z_r(R)=0$, then the each right ideal $A$ has a unique maximal essential extension $A^*$ in $R$. The set $L_r^*(R)(=L_r^*)$ of closed right ideals is a complete complemented modular lattice under the inclusion. If $\{C_i \mid i \in I\}$ is any collection of closed right ideals of $R$, then $\bigcup_{i\in I}^* C_i=(\sum_{i\in I} C_i)^*$. $(J_r^*; \cap, \cup)$ will denote the lattice of all annihilator right ideals of $R$. Then it is easily seen that $J_r^* \subseteqq L_r^*$. We note that the lattice $J_r^*$ is not usually a sublattice of $L_r^*$, although intersections are set-theoretic in both lattices. For convenience, we let $L_{r2}^*=L_r^* \cap L_2$ and $J_{r2}^*=J_r^* \cap L_2$, where $L_2$ is the set of two-sided ideals of $R$. Corresponding left properties of a ring $R$ are indicated by replacing each "$r$" by an "$l$". If $R$ is right locally uniform, then $L_r^*$ is an atomic lattice, and $A \in L_r^*$ is an atom if and only if $A$ is a closed uniform right ideal. Following R. E. Johnson we call

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*) Dedicated to Professor Keizo Asano for the celebration of his sixtieth birthday.

a ring $R$ a (*right*) *potent ring* (*P-ring*) if every nonzero closed right ideal of $R$ is potent. We say that uniform right ideals $A$ and $B$ are *similar* (in symbols; $A\sim B$) iff $A$ and $B$ contian mutually isomorphic nonzero right ideals $A'$ and $B'$ respectively. A ring $R$ said to be (*right*) *irreducible* iff $R$ is right locally uniform and $A\sim B$ for all uniform right ideals $A$ and $B$ of $R$. A right locally uniform irreducible ring with $Z_r(R)=0$ is called here an *I-ring*. An $I$-ring which is also a $P$-ring will be called a *PI-ring*. We note that a ring $R$ is a $PI$-ring if and only if $R$ is a $PI$-ring in the sense of R. E. Johnson [6]. A ring $R$ is said to be *residue-finite* if the following condition is satisfied:

The factor ring $R/T$ is finite dimensional as a right $R$-module for any nonzero $T\in L^*_{r2}$.

If $R$ is finite dimensional, then evidently $R$ is residue-finite. If $R$ is a prime ring, then $R$ is residue-finite, becauce $L^*_{r2}=\{0, R\}$. A $PI$-ring which is countably dimensional will be called a *CPI-ring*. Let $M$ be a right $R$-module. If $M$ is an $n$-dimensional in the sense of Goldie, then we write $n=\dim_R M$.

Concerning the terminologies we refer to [4] and [6].

**2. Residue-finite CPI-rings.**

**Theorem 1.** *If $R$ is a residue-finite CPI-ring, then the following properties hold:*

(1) $L^*_{r2}=J^*_{r2}=\{A^r \mid A\in L^*_r: atom\}$.

(2) *$L^*_{r2}$ is a chain and there exist the following two types:*

*(A)*: $R=T_0\supset T_1\supset T_2\supset\cdots$ *and* $\bigcup_{p=0}^{\infty} T_p=0$.

*(B)*: *There exists an integer $p$ such that* $R=T_0\supset T_1\supset T_2\supset\cdots\supset T_p\supset T_{p+1}=0$.

(3) *For each nonzero $T_p\in L^*_{r2}$, there exists an independent set $\{A_1,\cdots,A_n\}$ of atoms of $L^*_r$ such that* $A_1\cup^*\cdots\cup^* A_n\cup^* T_p=T_{p-1}$ *and* $(A_1\cup^*\cdots\cup^* A_n)\cap T_p=0$.

(4) *If $A$ is an atom of $L^*_r$, then $A\subseteq T_p$ and $A\not\subseteq T_{p+1}$ if and only if $A^r=T_{p+1}$.*

The lattices $J^*_r$ and $J^*_l$ are dual isomorpic under the corresponding $A\to A^l$, $A\in J^*_r$. Hence if $J^*_{r2}$ consists of $R=T_0\supset T_1\supset T_2\supset\cdots$, $\bigcap_{p=0}^{\infty} T_p=0$ or $R=T_0\supset T_1\supset T_2\supset\cdots\supset T_p\supset T_{p+1}=0$, then $J^*_{l2}$ consists of $0=T^l_0\subset T^l_1\subset T^l_2\subset\cdots$, $\bigcup_{p=0}^{\infty} T^l_p=R$ or $0=T^l_0\subset T^l_1\subset T^l_2\subset\cdots\subset T^l_{p+1}=R$, respectively.

**Lemma 1.** *Let $J^*_{l2}=\{T^l_0, T^l_1, T^l_2,\cdots\}$. Then:*

(1) *For each $T^l_p\neq R$, there exists a potent atom $B\in J^*_l$ such that $B\subseteq T^l_{p+1}$ and $B\cap T^l_p=0$.*

(2) *If $B$ is a potent atom of $J^*_l$, then $B\subseteq T^l_{p+1}$ and $B\not\subseteq T^l_p$ if and only if $B^l=T^l_p$.*

By [5], the lattice $J^*_l$ is an upper semi-modular lattice. Hence for each $B\in J^*_l$ such that the interval $[0, B]$ is a finite length, we can define, by Theorem 14 of [1], the dimension of $B$ as the maximal length of chains between 0 and $B$. If the dimension of $B$ is $n$, then we write $n=\dim B$.

**Lemma 2.** (1) $\dim_R(R/T_p)=d_p$ *if and only if* $\dim T_p^l=d_p$.

(2) *For each nonzero* $T_p$, *there exists an independent set* $\{B_i\}_{i=d_{p-1}+1}^{d_p}$ *of potent atoms of* $J_l^*$ *such that*

$T_p^l=T_{p-1}^l\cup(B_{d_{p-1}+1}\cup\cdots\cup B_{d_p})$ *and* $(B_{d_{p-1}+1}\cup\cdots\cup B_{d_p})\cap T_{p-1}^l=0$.

Let $\dim_R(R/T_p)=d_p$ for each nonzero $T_p\in L_{r2}^*$. Then evidently $\dim_R(T_{p-1}/T_p)=d_p-d_{p-1}$. If $R$ satisfies (A) in Theorem 1, we shall call the ring $R$ is of type (A), and $(d_1, d_2-d_1, \cdots, d_p-d_{p-1}, \cdots)$ is called *a set of block numbers of* $R$.

If $R$ satisfies (B) in Theorem 1, we shall call the ring $R$ is of type (B), and $(d_1, d_2-d_1, \cdots, d_p-d_{p-1}, \infty)$ is called a *set of block numbers of* $R$.

Let $R$ be a ring with $Z_r(R)=0$. As is well known the maximal right quotient ring $\hat{R}$ of $R$ is right $R$-injective and is a right self-injective (von Neumann) regular ring (see [2]). Let $L$ be an atomic lattice with 1. A set $\{a_i\}$ of atoms of $L$ is *independent* iff $a_i\cap(\bigcup_{j\neq i}a_j)=0$ for each $i$. An independent set $\{a_i\}$ of atoms of $L$ is called a *basis* of $L$ if $\bigcup_i a_i=1$.

In order to make further progress we need the following definition:

Let $R$ be a residue-finite *PI*- ring. $R$ is said to be *complemented* with respect to $L_{r2}^*$ if there exists a set $\{B_i\}$ of potent atoms of $J_l^*$ such that

(a) $\{B_i\}$ satisfies the condition (2) in Lemma 2, and

(b) For each nonzero $T_p$, $T_p\bigcup^* T_p^c=R$ and $T_p\cap T_p^c=0$, where $T_p^c=(\bigcup_{j>d_p}B_j)^r$. In addition, if $\bigcup_p^* T_p^c=R$, then $R$ is said to be *s-complemented* with respect to $L_{r2}^*$.

The following are examples of rings which are $s$-complemented with respect to $L_{r2}^*$.

(i) $R$ is an *FPI*-ring in the sense of [6].

(ii) Let $R$ be a residue-finite *CPI*-ring and let $\hat{R}$ be the maximal right quotient ring of $R$. If $\hat{R}$ is a left quotient ring of $R$, then $R$ is $s$-complemented with respect to $L_{r2}^*$ (see [7]).

(iii) Let $F$ be a division ring. If $A$ and $B$ are subsets of $F$, then we denote by $AB^{-1}$ the set $\{ab^{-1}\mid a\in A, b\in B, b\neq 0\}$. Let $\omega$ be the countable ordinal number. We denote by $(F)_\omega$ the ring of all column-finite $\omega\times\omega$ matrices over $F$. Let $F_{ij}$ be additive subgroups of $F$ such that $F_{ij}F_{jk}\subseteq F_{jk}(i,j,k=1,2,\cdots)$. Let $S=\{a\in(F)_\omega\mid a=(a_{ij}), a_{ij}\in F_{ij}\}$. Clearly $S$ is a subring of $R$. The ring $S$ will be called a $T$-ring (triangular-block matrix ring) with type (A) in $(F)_\omega$ iff there exist integers $0=d_0<d_1<d_2<\cdots<d_n<\cdots$ such that $F_{ij}\neq 0$ iff $i>d_p$ and $d_p<j\leqq d_{p+1}(p=0,1,2,\cdots)$. If $F_{11}F_{11}^{-1}=F$ and $F_{jj}F_{kj}^{-1}=F(2\leqq j<k)$, then $S$ is $s$-complemented with respect to $L_{r2}^*$ and a residue-finite *CPI*-ring

with type (A) (see [7], Theorem 2).

Let $R$ be $s$-complemented with respect to $L^*_{r2}$ with type (A) and let $\{B_i\}$ be potent atoms of $J^*_l$ which satisfies the conditions (a) and (b). Now we set $A_i=(\bigcup_{j\neq i} B_j)^r$. Then the following lemma holds.

**Lemma 3.** (1) *$\{A_i\}$ and $\{B_i\}$ are independent atoms of $L^*_r$ and $J^*_l$ respectively.*

(2) *For each $p$, $T_{p-1}=T_p\cup(A_{d_{p-1}+1}\cup\cdots\cup A_{d_p})$ and $T_p\cap(A_{d_{p-1}+1}\cup\cdots\cup A_{d_p})=0$.*

(3) $\bigcup^*_i A_i=R$.

(4) $B_i=(\bigcup_{j\neq i} A_j)^l$.

Now, we can summarize the above-mentioned results as follows:

**Theorem 2.** *Let $R$ be a CPI-ring with type* (A) *and let $(d_1, d_2, \cdots, d_n, \cdots)$ be the set of block numbers of $R$. If $R$ is s-complemented with respect to $L^*_{r2}$, then there exist potent atomic bases $\{B_1, B_2\cdots, B_n, \cdots\}$ for $J^*_l$ and $\{A_1, A_2, \cdots, A_n, \cdots\}$ for $L^*_r$ such that:*

(1) $A_i=(\bigcup_{j\neq i} B_j)^r$ *and* $B_i=(\bigcup_{j\neq i} A_j)^l$, $(i=1, 2, \cdots)$.

(2) $J^*_{r2}=L^*_{r2}=\{A^r_i\,|\,i=1, 2, \cdots\}$, $J^*_{l2}=\{B^l_i\,|\,i=1, 2, \cdots\}$.

(3) $A^r_1\geqq A^r_2\geqq\cdots\geqq A^r_n\geqq\cdots, \bigcup^\infty_{n=1} A^r_n=0$ *and* $0=B^l_1\leqq B^l_2\leqq\cdots\leqq B^l_n\leqq\cdots, \bigcup^\infty_{n=1} B^l_n=R$.

(4) $A^r_i=A^r_j$ *and* $B^l_i=B^l_j$ *iff* $d_0+d_1+\cdots+d_p<i$ *and* $j\leqq d_0+d_1+\cdots+d_{p+1}$ *for some $p$, where $d_0=0$.*

(5) $A_iB_j\neq 0$ *iff* $i>d_0+\cdots+d_p$ *and* $d_0+\cdots+d_p<j\leqq d_0+\cdots+d_{p+1}$ *for some $p$.*

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