227. *Wirtinger Presentations of Knot Groups**)

By Takeshi YAJIMA
Department of Mathematics, Osaka City University

(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 12, 1970)

In this note we shall give an algebraic proof of the following theorem, which is concerned with [2] and [3].

Theorem. *If a finitely presented group G satisfies the conditions* (a) $G/[G,G]$ *is isomorphic to a free abelian group of rank* $\mu \geqq 1$, (b) *the weight of G equals to* μ, (c) $H_2(G)=0$, *then G has Wirtinger presentations.*

1. Let E be an arbitrary subset of a group G. We shall denote by E^G the normal closure of E. If $E^G=G$ for some finite subset $E=(g_1,\cdots,g_n)$, then we shall call E a *nucleus* of G, and call n the order of the nucleus. Kervaire [2] called the minimal order of nuclei of G the weight of G.

The following proposition is obvious.

(1.1) Let $(g_1,\cdots,g_n)$ and $(h_1,\cdots,h_n)$ be n-tupels of G such that the transformation $(g_1,\cdots,g_n)\to(h_1,\cdots,h_n)$ is obtained by a finite sequence of transformations of the following types:

(i) $(g_1,\cdots,g_n)\to(g_1^{\varepsilon_1},\cdots,g_n^{\varepsilon_n})$, $\varepsilon_i=\pm 1$, $i=1,\cdots,n$,

(ii) $(g_1,\cdots,g_n)\to(g_{i_1},\cdots,g_{i_n})$, where $(i_1,\cdots,i_n)$ is a permutation of $(1,2,\cdots,n)$,

(iii) $(\cdots,g_i,\cdots,g_j,\cdots)\to(\cdots,g_i,\cdots,g_i^{\varepsilon}g_j,\cdots)$ or $(\cdots,g_i,\cdots,g_jg_i^{\varepsilon},\cdots)$, $\varepsilon=\pm 1$. Then $(h_1,\cdots,h_n)^G=(g_1,\cdots,g_n)^G$.

Let $(x_1,\cdots,x_n:r_1,\cdots,r_m)$ be a presentation of a group G. If each relator r_i is described in a form $x_i^{-1}w_{ij}x_jw_{ij}^{-1}$, i.e. $x_i=w_{ij}x_jw_{ij}^{-1}$ as a relation, then we call the presentation a *Wirtinger presentation* of G.

Let $F=F[x_1,\cdots,x_n]$ be a free group generated by free generators $x_1,\cdots,x_n$, and let R be the kernel $(r_1,\cdots,r_m)^F$ of the homomorphism $\varphi:F\to G$. Hopf [1] defined the second homology group $H_2(G)$ as the group $[F,F]\cap R/[F,R]$, and proved that it does not depend on the underlying free group F.

(1.2) Suppose a group G satisfies the condition (c) of the theorem and $(x_1,\cdots,x_n:r_1,\cdots,r_l,r_{l+1},\cdots,r_m)$ is a presentation of G. If $r_{l+1},\cdots,r_m\in[F,F]$, then G has also a presentation $G=(x_1,\cdots,x_n:r_1,\cdots,r_l,[r_i,x_j],i=l+1,\cdots,m,j=1,\cdots,n)$.

Proof. We shall prove that $(r_1,\cdots,r_m)^F=(r_1,\cdots,r_l,\{[r_i,x_j]\})^F$. $(r_1,\cdots,r_m)^F\supset(r_1,\cdots,r_l,\{[r_i,x_j]\})^F$ is trivial. Since $r_k\in[F,F]$ for k

*) Dedicated to Professor Keizo Asano on his 60th birthday.

$=l+1,\cdots,m$ and $[F,F]\cap R=[F,R]$, r_k is freely equivalent to a product $\prod_\lambda u_\lambda[r_{i_\lambda}, x_{j_\lambda}]^{\varepsilon_\lambda}u_\lambda^{-1}$, $u_\lambda\in F$, $\varepsilon_\lambda=\pm 1$. Therefore $r_k\in(r_1,\cdots,r_l,\{[r_i,x_j]\})^F$ for $k=l+1,\cdots,m$ and we have $(r_1,\cdots,r_m)^F\subset(r_1,\cdots,r_l,\{[r_i,x_j]\})^F$.

2. Suppose a group G satisfies the conditions (a) and (b) of the theorem, and ψ is the homomorphism $G\to G/[G,G]\cong Z(t_1)\times\cdots\times Z(t_\mu)$, where $Z(t_i)$ is an infinite cyclic group generated by t_i.

(2.1) If a group G satisfies the conditions (a) and (b) of the theorem, then there exists a nucleus $(h_1,\cdots,h_\mu)$ such that $\psi(h_i)=t_i$, $i=1,\cdots,\mu$.

Proof. There exists an ordered set of integers $(\nu_1,\cdots,\nu_\mu)$ such that $\psi(g)=t_1^{\nu_1}\cdots t_\mu^{\nu_\mu}$ for every element $g\in G$. We shall denote this set of integers by $\nu(g)$ and call ν_i the *i-th index* of g.

Let $(g_1,\cdots,g_\mu)$ be an arbitrary nucleus. Putting $(g_1^{(0)},\cdots,g_\mu^{(0)})=(g_1,\cdots,g_\mu)$, we shall construct $(g_1^{(i)},\cdots,g_\mu^{(i)})$ successively for $i=1,\cdots,\mu$ such that

$$(g_1^{(i)},\cdots,g_\mu^{(i)})^G=(g_1,\cdots,g_\mu)^G=G, \tag{1}$$

$$\begin{aligned}
\nu(g_1^{(i)})&=(1,0,\quad\cdots,0,\quad \nu_{1,i+1}^{(i)},\cdots,\nu_{1,\mu}^{(i)}),\\
\nu(g_2^{(i)})&=(0,1,0,\cdots,0,\quad \nu_{2,i+1}^{(i)},\cdots,\nu_{2,\mu}^{(i)}),\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\\
\nu(g_i^{(i)})&=(0,\quad\cdots,0,1,\quad \nu_{i,i+1}^{(i)},\cdots,\nu_{i,\mu}^{(i)}),\\
\nu(g_{i+1}^{(i)})&=(0,\quad\cdots,0,\quad \nu_{i+1,i+1}^{(i)},\cdots,\nu_{i+1,\mu}^{(i)}),\\
&\cdots\quad\cdots\quad\cdots\quad\cdots\\
\nu(g_\mu^{(i)})&=(0,\quad\cdots,0,\quad \nu_{\mu,i+1}^{(i)},\cdots,\nu_{\mu,\mu}^{(i)}).
\end{aligned} \tag{2}$$

Suppose $(g_1^{(i)},\cdots,g_\mu^{(i)})$ is constructed already. In virtue of (1.1), (i), we can assume that $\nu_{j,i+1}^{(i)}\geqq 0$ for $j=i+1,\cdots,\mu$. We assert that $\nu_{i+1,i+1}^{(i)},\cdots,\nu_{\mu,i+1}^{(i)}$ are not all zero, moreover that the greatest common divisor of $(\nu_{i+1,i+1}^{(i)},\cdots,\nu_{\mu,i+1}^{(i)})$ must be 1.

Since ψ is surjective, there exists an element $a_{i+1}\in G$ such that $\psi(a_{i+1})=t_{i+1}$ and that it has an expression $a_{i+1}=\prod_k u_k g_{j_k}^{(i)\varepsilon_k}u_k^{-1}$, where $u_k\in G$ and $\varepsilon_k=\pm 1$. Because the jth index of a_{i+1} is zero for $1\leqq j\leqq i$, the exponent sum of $g_j^{(i)}$ in the expression of a_{i+1} must be zero. Therefore $(i+1)$th indices of $g_1^{(i)},\cdots,g_i^{(i)}$ have no effect upon that of a_{i+1}. If $\nu_{j,i+1}^{(i)}=0$ for all $j=i+1,\cdots,\mu$ then the $(i+1)$th index of a_{i+1} is zero. This contradicts the assumption $\psi(a_{i+1})=t_{i+1}$. The same reason as the above guarantees that the greatest common divisor of $(\nu_{i+1,i+1}^{(i)},\cdots,\nu_{\mu,i+1}^{(i)})$ equals to 1.

Therefore, in virtue of (1.1), we can replace $(g_{i+1}^{(i)},\cdots,g_\mu^{(i)})$ by $(\bar{g}_{i+1}^{(i)},\cdots,\bar{g}_\mu^{(i)})$ such that the $(i+1)$th indices $(\bar{\nu}_{i+1,i+1}^{(i)},\cdots,\bar{\nu}_{\mu,i+1}^{(i)})$ of them equal to $(1,0,\cdots,0)$ and that $(g_1^{(i)},\cdots,g_i^{(i)},\bar{g}_{i+1}^{(i)},\cdots,\bar{g}_\mu^{(i)})^G=(g_1^{(i)},\cdots,g_\mu^{(i)})^G$. Then it is easy to get the required nucleus $(g_1^{(i+1)},\cdots,g_\mu^{(i+1)})$. This complete the proof of (2.1).

Let $(x_1,\cdots,x_\mu,c_1,\cdots,c_n:r_1,\cdots,r_m)$ be a presentation of a group

G, which satisfies the conditions (a) and (b). Let $F=F[x_1, \cdots, x_\mu, c_1, \cdots, c_n]$ and let φ, ψ be the same as the preceeding definitions. If the conditions

$$(\alpha) \quad \psi\varphi(x_i)=t_i, \qquad i=1, \cdots, \mu,$$
$$\psi\varphi(c_i)=1, \qquad i=1, \cdots, n,$$
$$(\beta) \quad (\varphi(x_1), \cdots, \varphi(x_\mu)) \text{ is a nucleus of } G,$$

are satisfied, then we call the presentation *a canonical presentation* of G.

In virtue of (2.1), we can easily verify the following proposition:

(2.2) If a group G satisfies the conditions (a) and (b), then G has canonical presentations.

Note that the exponent sum of each generator x_i in every relator r_j of a canonical presentation must be zero.

3. Let $P_0=(x_1, \cdots, x_\mu, c_1, \cdots, c_n : r_1, \cdots, r_m)$ be a canonical presentation of G. The definition of nucleus implies that $(r_1, \cdots, r_m)$ induces the following n relations:

$$s_\lambda : c_\lambda = \prod_{j=1}^{k_\lambda} w_{\lambda j} x_{i_{\lambda j}}^{\varepsilon_{\lambda j}} w_{\lambda j}^{-1}, \qquad \varepsilon_{\lambda j}=\pm 1, j=1, \cdots, k_\lambda, \lambda=1, \cdots, n,$$

where $w_{\lambda j}$ is a word of F. Denote the word $\prod_{j=1}^{k_\lambda} w_{\lambda j} x_{i_{\lambda j}}^{\varepsilon_{\lambda j}} w_{\lambda j}^{-1}$ by $C_\lambda(x, c)$ for $\lambda=1, \cdots, n$. Replace every c_λ in $r_1, \cdots, r_m$ by $C_\lambda(x, c)$ and denote the replaced relators by $r_1^*, \cdots, r_m^*$ respectively. Then we have, by Tietze transformation, the following presentation of G:

$$P_1=(x_1, \cdots, x_\mu, c_1, \cdots, c_n : r_1^*, \cdots, r_m^*, s_1, \cdots, s_n).$$

Since $\psi(c_\lambda)=1$, the exponent sum of each x_i, and also that of c_i in $C_\lambda(x, c)$ must be zero. From this fact and the notice in the preceeding section, it follows that $r_i^* \in [F, F]$.

Now add new generators $y_{\lambda j}$ and new relations

$$t_{\lambda j} : y_{\lambda j} = w_{\lambda j} x_{i_{\lambda j}} w_{\lambda j}^{-1}, \qquad j=1, \cdots, k_\lambda, \lambda=1, \cdots, n,$$

to the presentation P_1, and put $F'=F[x_1, \cdots, x_\mu, c_1, \cdots, c_n, \{y_{\lambda j}\}]$. Then,

$$P_2=(x_1, \cdots, x_\mu, c_1, \cdots, c_n, \{y_{\lambda j}\} : r_1^*, \cdots, r_m^*, s_1, \cdots, s_n, \{t_{\lambda j}\})$$

is a presentation of G. Note that $r_1^*, \cdots, r_m^*$ are still contained in $[F', F']$.

Now we shall eliminate generators $c_1, , \cdots, c_n$ from P_2. Replace every $w_{\lambda j} x_{i_{\lambda j}} w_{\lambda j}^{-1}$ in s_λ by $y_{\lambda j}$, and denote it by s_λ^* as follows:

$$s_\lambda^* : c_\lambda = \prod_{j=1}^{k_\lambda} y_{\lambda j}^{\varepsilon_{\lambda j}}, \qquad \lambda=1, \cdots, n.$$

Moreover, using s_λ^*, replace every c_λ in r_i^* and also in $t_{\lambda j}$ by y-symbols, and denote the corresponding new relators and relations by r_i^{**} and $t_{\lambda j}^*$ respectively. Then we have

$$P_3=(x_1, \cdots, x_\mu, c_1, \cdots, c_n, \{y_{\lambda j}\} : r_1^{**}, \cdots, r_m^{**}, s_1^*, \cdots, s_n^*, \{t_{\lambda j}^*\}),$$

as a presentation of G. In the relators and relations of P_3, c_λ is contained only in s_λ^*, as a definition of c_λ, for $\lambda=1, \cdots, n$. Therefore we can eliminate generators $c_1, \cdots, c_n$ and relations $s_1^*, \cdots, s_n^*$ from P_3.

Thus we have a presentation of G,

$$P_4=(x_1, \cdots, x_\mu, \{y_{\lambda j}\}: r_1^{**}, \cdots, r_m^{**}, \{t_{\lambda j}^*\}),$$

Put $F''=F[x_1,\cdots,x_\mu,\{y_{\lambda j}\}]$. Then every r_i^{**} is still contained in $[F'',F'']$. Therefore, by the lemma (1.2), we have a presentation

$$G=(x_1, \cdots, x_\mu, \{y_{\lambda j}\}: \{t_{\lambda j}^*\}, \{[r_i^{**}, x_k]\}, \{[r_i^{**}, y_{\lambda j}]\}).$$

Every relator or relation in the last presentation is of the Wirtinger type. This completes the proof of the theorem.

References

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