250. *On H-closedness and the Wallman H-closed Extensions. I*[*)]

By Chien WENJEN
California State College at Long Beach, U. S. A.

(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 12, 1970)

1. Introduction. A completely regular space is said to be pseudo-compact if every real-valued continuous function on it is bounded [7]. The pseudo-compactness is characterized by the existence of a cluster point for each sequence of open subsets of the space [see 6]. The question arises whether the existence of a cluster point for each net, instead of a sequence, of open sets is equivalent to some known topological property. We shall identify such property as H-closedness.

By a space we shall mean a Hausdorff space and a pseudo cover of a space, a collection of open sets whose closures cover the space. An H-closed space is defined as one which is closed in every space containing it and also one with finite pseudo subcover property, that is, every open cover has a finite pseudo subcover [2], [4]. H-closed spaces, first studied by Alexandrov and Urysohn [3] and also by M. H. Stone [12], play the same role to Hausdorff spaces in the theory of extensions as compact spaces (the Stone-Čech compactifications) to completely regular spaces. Alexandrov commented in [1]: "While the theory of bicompact extensions of completely regular spaces can be considered as being in its fundamental features already constructed, concerning H-closed extensions of Hausdorff spaces we have for the time being only isolated, though sometimes very interesting results."

One of the fundamental concepts in the theory of H-closed spaces is the "centered system of bases" originated by Alexandrov [8]. The central idea of the present note is the "pseudo finite intersection property" (PFIP), an analogue of finite intersection property and Alexandrov centered systems, and the presentation will parallel to the machinery for compact spaces (see [7], [11]). We call a family of non-vacuous sets centered if the intersection of any two sets in it belongs to it; and a family of non-vacuous sets in a topological space is said to have PFIP if the intersection of finitely many members of the family contains a non-vacuous open set. We will show that every space can be embedded in an H-closed space as a dense set, a construction similar

*) Presented to the Amer. Math. Soc. Aug., SS (1970). The work was done durnig Sabatcial Leave, spring (1970).

to the Wallman compactification of T_1 spaces [13]. As applications, we show that the Stone-Weierstrass approximation theorem holds for Hausdorff spaces and that a regular space is completely regular if the continuous functions on the space separate the points.

2. Basic properties. Theorem 1. *A space X is H-closed if and only if each family of closed subsets of X with PFIP has a nonvoid intersection.*

Proof. *Sufficiency.* Suppose that each family of closed subsets of X with PFIP has a nonvoid intersection and let $\{A_\alpha : \alpha \in I\}$ be an open cover of X. Then $\bigcap_\alpha (X-A_\alpha)=\emptyset$ and $\{X-A_\alpha : \alpha \in I\}$ is a family of closed sets without PFIP. There are $A_{\alpha_1},\cdots,a_{\alpha_n}$ such that $\bigcap_{i=1}^n (X-A_{\alpha_i})$ contains no nonempty open set and $X=\overline{X-\bigcap_{i=1}^n (X-A_{\alpha_i})}=\overline{\bigcup_{i=1}^n A_{\alpha_i}}=\bigcup_{i=1}^n \bar{A}_{\alpha_i}$. The H-closedness of X is proved.

Necessity. Assume that $\{C_\alpha : \alpha \in I\}$ is a family of closed sets in X with PFIP and void intersection. $X=X-\bigcap_\alpha C_\alpha=\bigcup_\alpha (X-C_\alpha)$ and $\{X-C_\alpha : \alpha \in I\}$ is an open cover of X. For any finite subset of the cover, $\{X-C_{\alpha_1},\cdots,X-C_{\alpha_n}\}$, $\bigcup_{i=1}^n (X-C_{\alpha_i})=X-\bigcap_{i=1}^n C_{\alpha_i}\subset X-P$ and $\bigcup_{i=1}^n \overline{X-C_{\alpha_i}}\subset X-P$, where P is a nonvoid open set contained in $\bigcap_{i=1}^n C_{\beta_i}$. Then the open cover $\{X-C_\alpha : \alpha \in I\}$ of X has no finite pseudo cover and X is not H-closed.

Corollary (Alexandrov). *A space X is H-closed if and only if $\bigcap_\alpha \bar{O}_\alpha \neq \emptyset$ for each family $\{O_\alpha\}$ of open subsets of X with PFIP.*

Proof. The necessity follows from theorem 1. In order to prove the sufficiency let $\{O_\alpha\}$ be the family of interiors of the closed sets $\{C_\alpha\}$ with PFIP. Then $\{O_\alpha\}$ has the PFIP and $\bigcap_\alpha \bar{O}_\alpha \neq \emptyset$ implies $\bigcap_\alpha C_\alpha \neq \emptyset$. X is H-closed by Theorem 1.

A point s of a space is called a cluster point of a net $\{O_\alpha ; \alpha \in I\}$ of open sets O_α if, for a given $\lambda \in I$ and a neighborhood N of s, there is an $\alpha_1 \in I$ such that $\alpha_1 > \lambda$ and N intersects O_{α_1} in a nonvacuous set.

Theorem 2. *A space X is H-closed if and only if each netof open sets in X has a cluster point.*

Proof. *Necessity.* Let $\{O_\alpha : \alpha \in I\}$ be a net of open sets in an H-closed space X and let $C_\lambda=\overline{\bigcup_{\alpha>\lambda} O_\alpha}\cdot\{C_\lambda ; \lambda \in I\}$ is a collection of closed sets with PFIP and $\bigcap_\lambda C_\lambda \neq \emptyset$ by Theorem 1. Any point belonging to $\bigcap_\lambda C_\lambda$ is a cluster point of the net $\{O_\alpha\}$.

Sufficiency. Let X be a Hausdorff space in which every net of open sets has a cluster point and let $\mathfrak{C}$ be a collection of closed subsets of X with PFIP. Define $\mathfrak{C}'$ to be the family of all finite intersections of members of $\mathfrak{C}$ and let $\mathfrak{C}'$ be directed by the inclusion relation. Then Φ

$=\{O_c : O_c = \text{interior of } C, C \in \mathfrak{C}'\}$ is a net of open sets directed by inclusion and has a cluster point s. If C_1 and C_2 belong to $\mathfrak{C}'$ and $C_1 \subset C_2$, then $O_{c_1} \subset O_{c_2}$ and the net $\{O_c\}$ is eventually in every open set $O \in \Phi$. Then s belongs to each C of $\mathfrak{C}$ and $\bigcap_{\alpha \in \mathfrak{C}} C$ is nonvid. The H-closedness of X is proved.

Theorem 3. *The Cartesian product X of H-closed spaces $X_\alpha (\alpha \in I)$ is H-closed relative to the product topology.*

Proof. Let $\mathfrak{B}$ be a family of subsets of X with PFIP. It is sufficient to show that $\bigcap \{\bar{B} : B \in \mathfrak{B}\}$ is nonvoid. We may assume by Zorn's or Tukey's lemma that $\mathfrak{B}$ is maximal with respect to the PFIP. If π_α is the projection operation from X into a coordinate space $X_\alpha \cdot \{\pi_\alpha B ; B \in \mathfrak{B}\}$ has the PFIP and $\bigcap \overline{\{\pi_\alpha B ; B \in \mathfrak{B}\}}$ is nonvoid. Choose $x_\alpha \in \bigcap_{B \in \mathfrak{B}} \pi_\alpha B$. The point x whose α-coordinate is x_α has each of its neighborhood intersecting each member of $\mathfrak{B}$ in a set which contains nonempty open set. Therefore x belongs to $\bar{B}$ for each $B \in \mathfrak{B}$ and the theorem is proved.

3. Pseudo-homeomorphism. Definition. Let X_1 and X_2 be two topological spaces and let f be a mapping of X_1 into X_2. f is said to be pseudo-continuous [θ-continuous] at a point $x_1 \in X_1$ if for any neighbourhood $N_2(x_2)$ of the point $x_2 = f(x_1)$ there exists a neighborhood $N_1(x_1)$ such that $f(N_1(x_1)) \subset \overline{N_2(x_2)} [f(\overline{N_1(x_1)}) \subset \overline{N_2(x_2)}]$.

A one-one mapping of X_1 onto X_2 which is pseudo-continuous [θ-continuous] in both directions is called a pseudo-homomorphism [θ-homeomorphism] between X_1 and X_2.

The concept of θ-continuity is due to Fomin [6] and the pseudo-continuity is also called weak-continuity [5].

Lemma 1. *If an H-closed space X is separated by the algebra $C(X)$ of all real-valued continuous functions on it, then there is a one-one correspondence between the maximal ideals of $C(X)$ and the points in X.*

Proof. Let M be a maximal ideal in $C(X)$. Suppose that to each point $x \in X$ there is a function $f \in M$ such that $f(x) \neq \emptyset$. Then f does not vanish on the closure of some neighborhood $G(x)$ of the point x. The H-closedness of X implies the existence of $G_1(x_1), \cdots, G_n(x_n)$ with the property $\overline{G_1(x_1)} \cup \cdots \cup \overline{G_n(x_n)} = X$. The function $f_1^2 + \cdots + f_n^2$ does not vanish on X and M is not a proper ideal. The contradiction proves the lemma.

Lemma 2. *Let X be an H-closed space separated by $C(X)$, G an open subset of X and x_0 a point of X not belonging to $\bar{G}$. Then there is a function $f \in C(X)$ such that $f(x) = 1$ for $x \in \bar{G}$ and $f(x_0) = 0$.*

Proof. For each $x_i \in \bar{G}$ there is a $f_i \in C(x)$ satisfying $f_i(x_i) = 2$ and $f_i(x_0) = 0$. $N(x_i) = \{x ; f_i(x) > 1\}$. We can find $N(x_1), \cdots, N(x_n)$ with

$\overline{N(x_i)} \cup \cdots \cup \overline{N(x_n)} \supset \bar{G}$. $(f_1 \vee \cdots \vee f_n) \wedge 1$ is the desired function f.

Lemma 3. *If X is an H-closed space separated by $C(X)$ and G_1, G_2 two open subsets of X with disjoint closures, then there is a function f in $C(X)$ such that $f(x)=1$ on $\bar{G}_1$ and on $\bar{G}_2$.*

Proof. We can find n functions $f_1, \cdots, f_n \in C(X)$ such that $f_i(x) =2$ on $\bar{G}_1 f_i(x_i)=0, x_i \in \bar{G}_2$ and $\bigcup_{i=1}^{n} \overline{N_i(x_i)} = \bigcup_{i=1}^{n} \overline{\{x\,; f_i(x)<1, x \in X\}} \supset \bar{G}_2$. Then $f=\{(f_1-1) \wedge \cdots \wedge (f_n-1)\} \vee 0$.

Lemma 4. *If $C(X_1), C(X_2)$ separate H-closed spaces X_1, X_2 respectively, then the concepts of pseudo-continuity and θ-continuity between X_1 and X_2 coincide.*

Proof. Let f be a pseudo-continuous mapping from X_1 into X_2. To any open neighborhood G_2 of $X_2=f(X_1)$ there is an open neighborhood G_1 of x_1 such that $f(G_1) \subset \bar{G}_2$. Suppose $f(\bar{G}_1) \not\subset \bar{G}_2$. There is $g_1 \in \bar{G}_1 - G_1$ with the property $f(g_1) \notin \bar{G}_2$. If N_2 is an open neighborhood of $f(g_1)$ such that $\bar{N}_2 \cap \bar{G}_2 = \emptyset$, by Lemma 2 there is an open neighborhood N_1 of g_1 with $f(N_1) \subset \bar{N}_2$ This is impossible on account of $N_1 \cap G_1 \neq \emptyset$ and $f(N_1 \cap G_1) \subset \bar{G}_2$.

Theorem 4. *Let X_1, X_2 be two H-closed spaces separated by $C(X_1)$, $C(X_2)$ respectively. Then X_1 and X_2 are pseudo-homeomorphic if and only if $C(X_1)$ and $C(X_2)$ are algebraically isomorphic.*

Proof. If $C(X_1)$ and $C(X_2)$ are algebraically isomorphic, the one-one correspondence ρ between X_1 and X_2 is determined by the correspondence between the maximal ideals of $C(X_1)$ and $C(X_2)$. Let N_2 be an open neighborhood of $x_2 \in X_2$ and $\rho(x_1)=x_2$ $(X_2-\bar{N}_2 \neq \emptyset)$. It is sufficient to prove that there is an open neighborhood N_1 of x_1 with $\rho(\bar{N}_1) \subset \bar{N}_2$. We can find $f_2 \in C(X_2)$ such that $f_2(x)=1$ for $x \in \overline{X_2-\bar{N}_2}$ and $f_2(x_2)=0$ by Lemma 2. For $f_1 \in C(x_1)$ corresponding to f_2 the open set $M_1=\{x : f_1(x)<\frac{1}{2}, x \in X_1\}$ is mapped onto a set $\rho(M_1)=M_2=\{x : f_2(x)<\frac{1}{2}, x \in X_2\} \subset \bar{N}_2$. The theorem is proved.

Theorem 5. *Every H-closed space X separated by $C(X)$ is pseudo-homeomorphic to a compact space.*

Proof. Let $F(X)$ be the set of all continuous functions on X to $[0, 1]$, and let e be the evaluation map carrying x, X into $e(x) \in I^{F(x)}$, the product space of the unit interval I taken $F(X)$ times, whose f-th coordinate is $f(x)$ for each $f \in F(X)$. Then e maps X onto a subspace $e(X)$ of $I^{F(X)}$, whose closure $\overline{e(X)}$ in $I^{F(X)}$ is a compact set. A continuous function on $\overline{e(X)}$ is continuous on X and it follows from the construction of $e(X)$ that any function continuous on X is also continuous on $\overline{e(X)}$. Then $C(X)$ and $C(\overline{e(X)})$ are algebraically isomorphic and the theorem follows from Theorem 4.

References

[1] P. S. Alexandrov: Some results in the theory of topological spaces, obtained within the last twenty-five years. Uspekhi Math. Nauk, **15**, 25–95 (1960) (Russian Math. Surveys, **15**(3), 23–83 (1960)).

[2] P. S. Alexandrov and H. Hopf: Topologie. I. Berlin (1935).

[3] P. S. Alexandrov and P. Urysohn: Memoire sur les espaces topologiques compacts. Verh. Akad. Wetensch. Amsterdam, **14**, 1–96 (1929).

[4] N. Bourbaki: Topologie Generale, 4th ed. Actualities Sci. Ind. No. 1142, Hermann, Paris (1965).

[5] J. Flachsmeyer: *H*-algeschlossene Raume als schwash-stetige Bilder kompakter Raume. Math. Zeitschr., **91**, 336–343 (1966).

[6] S. Fomin: Extension of topological spaces. Ann. of Math., **44**, 471–480 (1943).

[7] L. Gillman and M. Jerison: Rings of Continuous Functions. Van Nostrand, New York (1960).

[8] S. Iliades and S. Fomin: The method of centered systems in the theory of topological spaces. Uspekhi Mat. Nauk, **21** (4), 47–76 (1966) (Russian Math. Surveys, **21**(4), 37–62 (1966).

[9] K. Iséki: *AU*-covering and compactness. Proc. Japan Acad., **33**, 363–367 (1957).

[10] M. Katétov: Über *H*-abgeschlossene und bikompakte Raume. Casopis Pest Mat. Fys., **69**, 36–49 (1940).

[11] J. L. Kelley: General Toplogy. Van Nostrand, New York (1955).

[12] M. H. Stone: Applications of the theory of Boolean rings to general topology. Trans. Amer. Math. Soc., **41**, 375–481 (1937).

[13] H. Wallman: Lattices and topological spaces. Ann. of Math., **42**(2), 687–697 (1941).