245. A Note on Compactness and Metacompactness

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The purpose of this paper is to give a short proof to a theorem by G. Aquaro [2] which states that a space X is compact if and only if it is countably compact and each open cover of X has a point countable open refinement. Arens and Dujundji used Zorn's lemma to prove that a T_1 space is compact if and only if it is metacompact and countably compact [1].

An open cover of a space X is said to be a point (finite) countable open cover of X if each point in the space is in only a (finite) countable number of members of the open cover [3].

Theorem 1. *If $C=\{G_\alpha \mid \alpha \in A\}$ is a point (finite) countable open cover of a space X which has no (finite) countable subcover, then there exists a sequence $\{x_n\}$ such that if $i \neq j$ then $\{x_i, x_j\} \cap G_\alpha = \phi$ for each $\alpha \in A$.*

Proof. Let $G_{(\alpha,1)} \in C$ and let $x_1 \in G_{(\alpha,1)}$. There exists $x_2 \in X - \cup\{G_\alpha \mid G_\alpha \in C$ and $x_1 \in G_\alpha\}$. For each positive integer n there exists $x_n \in X - \cup\{G_\alpha \mid x_i \in G_\alpha$ for some $i<n\}$. The constructed sequence $\{x_n\}$ is a desired sequence for the theorem.

Remark. It is immediate that the sequence in Theorem 1 has no cluster point. If it did then the cluster point would be in some $G_\alpha \in C$ which would result in a contradiction.

By the fact that a space X is countably compact if and only if each sequence has a cluster we now have the following corollaries. The first corollary was the key result in a paper by G. Aquaro [2].

Corollary 1. *If X is a countably compact topological space then for each point countable open covering of X there exists a finite subcovering.*

Corollary 2. *A space X is compact if and only if it is countably compact and each open cover of X has a point countable open refinement.*

The following corollary with the added condition of T_1 was first proven by Arens and Dujundji [1].

Corollary 3. *A space X is compact if and only if it is metacompact and countably compact.*

References

[1] R. Arens and J. Dugundji: Remarks on the concept of compactness. Portug. Math., **9**, 141–143 (1950).

[2] G. Aquaro: Point Countable Open Coverings in Countably Compact Spaces. General Topology and its Relations to Modern Analysis and Algebra. II. Proceedings of the Second Prague Topological Symposium (1966). Academia Publishing House of the Czechoslovak Academy of Sciences, 39–41.

[3] J. Greever: On some generalized compactness properties. Publ. Res. Inst. Math. Sci. Kyoto Univ. Ser. A, **4**, 39–49 (1968).