54. *Functional Dimension of Tensor Product*

By Shigeo TAKENAKA

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§1. Introduction. The purpose of this paper is to give a proof to the fact that the functional dimension of the tensor product of two topological vetcor spaces is equal to the sum of their functional dimensions.

A. N. Kolmogorov [1] showed that the asymptotic behavior of number of elements of a minimal ε-net of a totally bounded subset in a topological vector space plays the role of dimension of the space. He [2] also introduced the notions of the approximative dimension and the functional dimension of topological vector spaces. The functional dimension is not trivial for σ-Hilbert nuclear spaces as is shown in I. M. Gel'fand's book [3].

In this paper we modify the definition of the functional dimension d_f of σ-Hilbert nuclear spaces to the number which is equal to the functional dimension (defined by Kolmogorov) minus 1, and we prove the following theorem:

Theorem. *Let E_1 and E_2 be σ-Hilbert nuclear spaces. Then*

$$d_f(E_1 \otimes E_2) = d_f(E_1) + d_f(E_2).$$

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§2. Notations. We follow notations used by Kolmogorov [4]. Let E be a topological vector space, K be a totally bounded subset of E and S be its convex absorbing and barrelled neighbourhood of 0 in E. Then we call *ε-entropy* $H_\varepsilon(S, K)$ of K (with respect to S) the infimum of logarithm of number of *ε-nets* of K (with respect to S); that is,

$$H_\varepsilon(S, K) = \inf\{\log(\# N); N \subset E, \forall k \in K, \exists n \in N, k \in n + \varepsilon S\}.$$

We use the following notations for infinitesimals: $f(x) \succcurlyeq g(x)$ means $\lim_{x\to\infty} g(x)/f(x) < +\infty$; $f(x) \asymp g(x)$ means $f(x) \preccurlyeq g(x)$ and $f(x) \succcurlyeq g(x)$; $f(x) = \Omega(g(x))$ means $\lim_{x\to\infty} (f(x))^n / g(x) = 0$.

In this paper the notation log stands for the logarithm with respect to the base 2.

§3. Theorem of Mityagin and σ-Hilbert nuclear spaces. We define as follows: The set $\mathcal{E}$ is called *$\{a_n\}$-ellipsoid* when $\mathcal{E} = \{(\xi_n) \in (l^2); \sum_n |\xi_n a_n|^2 \leqq 1\}$, where $\{a_n\}$ is a monotonous increasing series of such numbers a_n that $a_n \geqq 1$ and $\lim_{n\to\infty} a_n = \infty$; the function $m(t)$ is defined by the formula $m(t) = \sup\{n; a_n \leqq t\}$; let S be the unit ball in (l^2).

Then the following theorem holds.

Theorem A (Mityagin [5]). *If $\mathcal{E}$ is an $\{a_n\}$-ellipsoid, then*

$$m\left(\frac{2}{\varepsilon}\right)\log\frac{4e}{\varepsilon}\geqq H_\varepsilon(S,\mathcal{E})\geqq\log(\mathrm{e})\int_1^{1/2\varepsilon}\frac{m(t)}{t}dt.$$

Now let

$$\gamma(\mathcal{E})=\lim_{\varepsilon\to 0}\frac{\log H_\varepsilon(S,\mathcal{E}))}{\log\log\frac{1}{\varepsilon}}-1,$$

then immediately from Theorem A we get

Proposition 1. *For sufficiently small positive* ε,

$$\frac{\log\left(m\left(\frac{2}{\varepsilon}\right)\right)}{\log\log\frac{1}{\varepsilon}}\succcurlyeq\gamma(\mathcal{E})\succcurlyeq\frac{\log\left(\int_1^{1/2\varepsilon}m(t)\frac{dt}{t}\right)}{\log\log\frac{1}{\varepsilon}}-1.$$

Theorem B (Gel'fand [3]). *Let E be a σ-Hilbert space and $U_n=\{x\in E\,;\,p_n(x)\leqq 1\}$, where p_n are its countable norms, then E is nuclear if and only if*

$$\sup_m\inf_n\lim_{\varepsilon\to 0}\frac{\log H_\varepsilon(U_m,U_n)}{\log\frac{1}{\varepsilon}}=0.$$

Following Kolmogorov [4] we define functional dimension $d_f(E)$ of a Frechét space E as follows:

$$d_f(E)=\sup_U\inf_V\lim_{\varepsilon\to 0}\frac{\log H_\varepsilon(U,V)}{\log\log\frac{1}{\varepsilon}}-1,$$

where U,V are convex barrelled and absorbing neighbourhood of 0 of E.

From Theorem B, we can consider that this functional dimension plays the role of dimensionality of σ-Hilbert spaces. Clearly, the tensor product of two σ-Hilbert (nuclear) spaces is also a σ-Hilbert (nuclear) space. In the following sections we shall show that the tensor product of two σ-Hilbert spaces with finite functional dimension has also finite functional dimension.

§4. The function $m(t)$ of an ellipsoid with finite γ. Let E be a σ-Hilbert nuclear space and p_n be its norms, E_n be the completion of E with respect to p_n. In E_n, $U_m=\{p_m(x)\leqq 1\}$ is an ellipsoid if $m>n$. If E has finite functional dimension, for arbitrary n there exists m such that

$$\lim_{\varepsilon\to 0}\frac{\log H_\varepsilon(U_n,U_m)}{\log\log\frac{1}{\varepsilon}}<\infty.$$

We shall characterize an ellipsoid of this type by the growth of $m(t)$.

Proposition 2. *Let $\mathcal{E}$ be an ellipsoid and $m(t)$ be the corresponding function defined in* § 3. *If $\gamma(\mathcal{E})=\beta$, then*

$$m\left(\frac{1}{\varepsilon}\right)\asymp\left(\log\frac{1}{\varepsilon}\right)^{\beta}\left(1+\Omega\left(\log\frac{1}{\varepsilon}\right)\right).$$

Proof. By Proposition 1, we have

$$m\left(\frac{2}{\varepsilon}\right)\succcurlyeq\left(\log\frac{1}{\varepsilon}\right)^{\beta}\left(1+\Omega\left(\log\frac{1}{\varepsilon}\right)\right),$$

$$\int_1^{1/2\varepsilon}\frac{m(t)}{t}dt\preccurlyeq\left(\log\frac{1}{\varepsilon}\right)^{\beta+1}\left(1+\Omega\left(\log\frac{1}{\varepsilon}\right)\right).$$

Now put, for positive α,

$$m\left(\frac{2}{\varepsilon}\right)\asymp\left(\log\frac{1}{\varepsilon}\right)^{\beta+\alpha+1}\left(1+\Omega\left(\log\frac{1}{\varepsilon}\right)\right);$$

then $\int_1^{1/2\varepsilon}\frac{m(t)}{t}dt\asymp\left(\log\frac{1}{\varepsilon}\right)^{\beta+\alpha+1}\left(1+\Omega\left(\log\frac{1}{\varepsilon}\right)\right).$

Therefore $\alpha=0$. And we have

$$m\left(\frac{1}{\varepsilon}\right)\asymp\left(\log\frac{1}{\varepsilon}\right)^{\beta}\left(1+\Omega\left(\log\frac{1}{\varepsilon}\right)\right). \qquad \text{Q.E.D.}$$

§ 5. Tensor product of two ellipsoids and its $m(t)$. Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces, and $\mathcal{E}_1=\{\sum_n|a_n\xi_n|^2\leqq 1\}$ and $\mathcal{E}_2=\{\sum_n|b_n\eta_n|^2\leqq 1\}$ be ellipsoids in $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Then $\mathcal{E}=\mathcal{E}_1\otimes\mathcal{E}_2$ is also an $\{c_{nm}\}$-*ellipsoid* in $\mathfrak{H}_1\otimes\mathfrak{H}_2$, where $c_{nm}=a_nb_m$. Let $m_1(t)$ and $m_2(t)$ be the functions corresponding to $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively, and suppose

$$m_1(t)\asymp(\log t)^{\alpha}(1+\Omega(\log t))$$

and

$$m_2(t)\asymp(\log t)^{\beta}(1+\Omega(\log t)).$$

Then $\frac{dm_1(t)}{dt}\Delta t$ and $\frac{dm_2(t)}{dt}\Delta t$ are numbers of axes whose lengths fall between t and $t+\Delta t$.

Now we estimate $m(t)$ of ε. We have

$$\int_1^{t-\Delta}\int_1^{(t-\Delta)/x}m_1'(x)m_2'(y)dxdy\preccurlyeq m(t)\preccurlyeq\int_1^{t+\Delta}\int_1^{(t+\Delta)/x}m_1'(x)m_2'(y)dxdy,$$

where $0\leqq\Delta\ll t$;

$$\begin{aligned}m(t)&\preccurlyeq\int_1^{t+\Delta}\alpha(\log x)^{\alpha-1}(1+\Omega(\log x))(\log(t+\Delta)-\log x)^{\beta}\frac{dx}{x}\\&\preccurlyeq\int_1^{t+\Delta}\alpha(\log(t+\Delta))^{\beta}(\log x)^{\alpha-1}(1+\Omega(\log t))\frac{dx}{x}\\&\preccurlyeq(\log(t+\Delta))^{\alpha+\beta}(1+\Omega(\log t))\\&\asymp(\log t)^{\alpha+\beta}(1+\Omega(\log t)).\end{aligned}$$

And we have

$$\begin{aligned}m(t)&\succcurlyeq\int_1^{\sqrt{t-\Delta}}\alpha(\log t)^{\alpha-1}(1+\Omega(\log t))\frac{\log(t-\Delta)}{2}\frac{dx}{x}\\&\asymp\frac{1}{2^{\alpha+\beta}}(\log(t-\Delta))^{\alpha+\beta}(1+\Omega(\log t))\end{aligned}$$

$\asymp (\log t)^{\alpha+\beta}(1+\Omega(\log t)).$ Q.E.D.

Thus we have

Proposition 3. $m(t) \asymp m_1(t) \cdot m_2(t)$.

§6. Functional dimension of tensor product of nuclear spaces.

Theorem. *Let E and F be σ-Hilbert nuclear spaces. Then*

$$d_f(E\otimes F)=d_f(E)+d_f(F).$$

Proof. Let $\{U_i\}$ and $\{V_j\}$ be fundamental neighbourhoods of E and F, respectively. Let

$$\gamma_n^m=\lim_{\varepsilon\to 0}\frac{\log H_\varepsilon(U_m, U_n)}{\log\log\frac{1}{\varepsilon}}-1,$$

and

$$\tilde{\gamma}_n^m=\lim_{\varepsilon\to 0}\frac{\log H_\varepsilon(V_m, V_n)}{\log\log\frac{1}{\varepsilon}}-1,$$

then $\sup_m \inf_n \gamma_n^m = d_f(E)$ and $\sup_m \inf_n \tilde{\gamma}_n^m = d_f(F)$. Put

$$\gamma_{nl}^{mk}=\lim_{\varepsilon\to 0}\frac{\log H_\varepsilon(U_m\otimes V_k, U_m\otimes V_l)}{\log\log\frac{1}{\varepsilon}}-1,$$

then $\sup_{m,k}\inf_{n,l}\gamma_{nl}^{mk}=d_f(E\otimes F)$. By Proposition 3, $\gamma_{nl}^{mk}=\gamma_n^m+\tilde{\gamma}_l^k$. Hence we get

$$d_f(E\otimes F)=\sup_{m,k}\inf_{n,l}\gamma_{nl}^{mk}=\sup_m\inf_n\gamma_n^m+\sup_k\inf_l\tilde{\gamma}_l^k$$
$$=d_f(E)+d_f(F).$$ Q.E.D.

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