146. *On a Mean Value Theorem for the Remainder Term in the Prime Number Theorem for Short Arithmetic Progressions*

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1. In 1965 Bombieri [1] improved almost ultimately the large sieve of Linnik and Rényi, and as an application he derived an astounding result on the average size of the remainder term in the prime number theorem for arithmetic progressions. Recently Jutila [4] has proved an analogous result for short arithmetic progressions, i.e. he considered the estimation of the expression

$$D(Q; x, h) = \sum_{q \le Q} \max_{(l,q)=1} \left| \psi(x+h; q, l) - \psi(x; q, l) - \frac{h}{\varphi(q)} \right|, \tag{1.1}$$

where $\psi(x; q, l)$ is the usual Čebyšev function for the arithmetic progression $\equiv l \bmod q$ and Q, h are appropriate functions of x.

Both results of Bombieri and Jutila have been obtained by reducing the problem to the estimation of the total density of zeros of 'many' L-functions. But Gallagher [2] has found a way to prove Bombieri's result without using the density theorem. In [4] an opinion is expressed that it seems difficult to prove a non-trivial estimation of (1.1) on the similar line. The purpose of the present paper is to offer such a proof.

Our main tool is the following beautiful inequality of Gallagher [3]: If

$$\sum_n n|a_n|^2 < +\infty,$$

then we have

$$\sum_{q \le Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{-T}^{T} \left| \sum_n a_n \chi(n) n^{-it} \right|^2 dt \ll \sum_n (Q^2 T + n)|a_n|^2, \tag{1.2}$$

where $\sum^*$ denotes a sum over all primitive characters mod q.

2. Before entering into the proof we list up here some definitions. Let $\Lambda(n)$ be the von Mangold function and let

$$\psi(x, \chi) = \sum_{n \le x} \chi(n) \Lambda(n).$$

Let $\mu(n)$ be the Möbius function and let

$$H(s, \chi) = \sum_{n \le Q^2 T} \mu(n) \chi(n) n^{-s}.$$

Let M be the number defined by

$$M=\max_{q\le Q}\max_{\chi \bmod q}\max_{|t|\le T}\left|L\left(\frac{1}{2}+it,\chi\right)\right|.$$

Further we restrict the size of Q^2T by $\log Q^2T\ll\log x$, and we set $g(s;x)=((x+h)^s-x^s)/s$.

3. It is easy to see that

$$D(Q;x,h)\ll\log^2 x\sum_{1<q\le Q}\frac{1}{q}\sideset{}{^*}\sum_{\chi \bmod q}|\psi(x+h,\chi)-\psi(x,\chi)| + \log x\left|\sum_{x\le n\le x+h}\Lambda(n)-h\right|+Q(\log x)^2. \tag{3.1}$$

Following [2] we have the integral expression

$$\begin{aligned}\psi(x+h,\chi)-\psi(x,\chi)&+0\left(\frac{x}{T}(\log x)^2+\log x\right)\\ &=-\frac{1}{2\pi i}\int_{\alpha-iT}^{\alpha+iT}(1-L(s,\chi)H(s,\chi))^2\frac{L'}{L}(s,\chi)g(s;x)ds\\ &\quad+\frac{1}{2\pi i}\left\{\int_{\beta-iT}^{\beta+iT}+\int_{\beta+iT}^{\alpha+iT}-\int_{\beta-iT}^{\alpha-iT}\right\}P(s,\chi)g(s;x)ds\\ &=I_1(\chi)+I_2(\chi)+I_3(\chi)+I_4(\chi),\ \text{say},\end{aligned} \tag{3.2}$$

where $\alpha=1+(\log x)^{-1}$, $\beta=1/2+2(\log x)^{-1}$ and

$$P(s,\chi)=L(s,\chi)L'(s,\chi)H^2(s,\chi)-2L'(s,\chi)H(s,\chi).$$

4. The sum

$$\sum_{1<q\le Q}\frac{1}{q}\sideset{}{^*}\sum_{\chi \bmod q}|I_2(\chi)| \tag{4.1}$$

does not exceed the constant multiple of

$$\begin{aligned}&M^2hx^{-1/2}\log x\int_1^Q\frac{dy}{y^2}\sum_{q\le y}\sideset{}{^*}\sum_{\chi \bmod q}\int_{\beta-iT}^{\beta+iT}|H(s,\chi)|^2dt\\ &+hx^{-1/2}\left\{\int_1^Q\frac{dy}{y^2}\sum_{q\le y}\sideset{}{^*}\sum_{\chi \bmod q}\int_{\beta-iT}^{\beta+iT}|L'(s,\chi)|^2dt\right\}^{1/2}\\ &\times\left\{\int_1^Q\frac{dy}{y^2}\sum_{q\le y}\sideset{}{^*}\sum_{\chi \bmod q}\int_{\beta-iT}^{\beta+iT}|H(s,\chi)|^2dt\right\}^{1/2}.\end{aligned}$$

Here we apply the inequality (1.2) to the integral involving $H(s,\chi)$ and we get

$$\begin{aligned}&\int_1^Q\frac{dy}{y^2}\sum_{q\le y}\sideset{}{^*}\sum_{\chi \bmod q}\int_{\beta-iT}^{\beta+iT}|H(s,\chi)|^2dt\\ &\ll\int_1^Q\frac{dy}{y^2}\sum_{n\le Q^2T}(y^2T+n)n^{-2\beta}\ll Q^2T\log x.\end{aligned}$$

This gives that the sum (4.1) is

$$\ll M^2Q^2Thx^{-1/2}(\log x)^2. \tag{4.2}$$

5. We have by the convexity argument

$$|L(s,\chi)|\ll M^{2(1-\sigma)}\log x,\qquad |L'(s,\chi)|\ll M^{2(1-\sigma)}\log^2 x$$

uniformly for $1/2\le\sigma\le\alpha$. Hence we have

$$\sum_{1<q\le Q}\frac{1}{q}\sideset{}{^*}\sum_{\chi \bmod q}|I_3(\chi)|$$

$$(5.1)\qquad \ll \frac{x}{T}(\log x)^3 \int_\beta^\alpha \left(\frac{M^4}{x}\right)^{1-\sigma} d\sigma \int_1^Q \frac{dy}{y^2} \sum_{q\le y} \sideset{}{^*}\sum_{\chi \bmod q} |H(\sigma+iT,\chi)|^2$$

$$+\frac{xQ^{1/2}}{T}(\log x)^2 \int_\beta^\alpha \left(\frac{M^2}{x}\right)^{1-\sigma} d\sigma \left\{\int_1^Q \frac{dy}{y^2} \sum_{q\le y} \sideset{}{^*}\sum_{\chi \bmod q} |H(\sigma+iT,\chi)|^2\right\}^{1/2}.$$

By the large sieve of Bombieri [1] we have

$$\int_1^Q \frac{dy}{y^2} \sum_{q\le y} \sideset{}{^*}\sum_{\chi \bmod q} |H(\sigma+iT,\chi)|^2 \ll \{Q+(Q^2T)^{2(1-\sigma)}\}(\log x)^2.$$

Hence we see that the left side of (5.1) is

$$(5.2)\qquad \ll \frac{Qx}{T}(\log x)^5 \int_\beta^\alpha \left(\frac{M^4Q^4T^2}{x}\right)^{1-\sigma} d\sigma.$$

Obviously we have the same estimation for the sum where $I_3(\chi)$ is replaced by $I_4(\chi)$.

6. Now let consider the sum

$$\sum_{1<q\le Q} \frac{1}{q} \sideset{}{^*}\sum_{\chi \bmod q} |I_1(\chi)|.$$

We divide this into two parts R_1 and R_2 according to $1<q\le(\log x)^K$ and $(\log x)^K<q\le Q$ respectively, where K is taken to be sufficiently large.

With the aid of (1.2) we have

$$(6.1)\qquad \begin{aligned} R^2 &\ll h(\log x)^{2-K} \sum_{q\le Q} \sideset{}{^*}\sum_{\chi \bmod q} \int_{\alpha-iT}^{\alpha+iT} |1-L(s,\chi)H(s,\chi)|^2\,dt \\ &\ll h(\log x)^{2-K} \sum_{Q^2T\le n} (Q^2T+n)d^2(n)n^{-2\alpha} \\ &\ll h(\log x)^{6-K}, \end{aligned}$$

where $d(n)$ is the number of divisors of n.

In order to estimate R_1 we appeal to Siegel's theorem [Satz 8.1 of Chap. 4; 5] and Tatuzawa's theorem [Satz 6.2 of Chap. 8; ibid] from which we have, on the condition $1<q\le(\log x)^K$,

$$\left|\frac{L'}{L}(s,\chi)\right| \ll (\log x)^2$$

uniformly for $\sigma \ge 1-(\log x)^{-4/5}=\eta$, $|t|\le T$. Hence we have

$$(6.2)\qquad \begin{aligned} I_1(\chi) = -\frac{1}{2\pi i}&\left\{\int_{\eta-iT}^{\eta+iT} + \int_{\eta+iT}^{\alpha+iT} - \int_{\eta-iT}^{\alpha-iT}\right\} \\ &\times(1-L(s,\chi)H(s,\chi))^2 \frac{L'}{L}(s,\chi)g(s;x)ds. \end{aligned}$$

Here we have

$$\int_{1/2-iT}^{1/2+iT} |1-L(s,\chi)H(s,\chi)|^2 dt \ll M^2Q^2T$$

and

$$\int_{\alpha-iT}^{\alpha+iT} |1-L(s,\chi)H(s,\chi)|^2\,dt \ll (\log x)^4.$$

From these and by the convexity argument, the first integral of (6.2) is

$$(6.3)\qquad \ll h(\log x)^6\left(\frac{M^4Q^4T^2}{x}\right)^{1-\eta}.$$

And again by the convexity argument the second and third integrals of (6.2) are

$$(6.4)\qquad \ll \frac{x}{T}(\log x)^6\int_\eta^\alpha\left(\frac{M^4Q^4T^2}{x}\right)^{1-\sigma}d\sigma.$$

From (6.2), (6.3) and (6.4) we have

$$(6.5)\qquad R_1\ll h(\log x)^{6+K}\left(\frac{M^4Q^4T^2}{x}\right)^{1-\eta}+\frac{x}{T}(\log x)^{6+K}\int_\eta^\alpha\left(\frac{M^4Q^4T^2}{x}\right)^{1-\sigma}d\sigma.$$

7. Now let ε be an arbitrarily small positive constant and let

$$x^{1-\varepsilon}=M^4Q^4T^2,\qquad T=Qx^{\theta}(\log x)^K.$$

Then we have collecting (3.1), (3.2), (4.2), (5.2), (6.1) and (6.5)

$$(7.1)\qquad \begin{aligned} D(Q;x,h)\ll h(\log x)^{8-K}+x^{1-\theta}(\log x)^{4-K} \\ +\log x\left|\sum_{x\le n\le x+h}\Lambda(n)-h\right|. \end{aligned}$$

Here we can take

$$M=Q^{1/2}T^{1/6}(\log x)^2$$

(see [§ 4 of Chap. 9; 5]). And if $h\gg x^{5/8+\varepsilon}$, then it is well-known that the last term of (7.1) is

$$(7.2)\qquad \ll h(\log x)^{-A}$$

for arbitrarily large A. This has been proved by the density theorem of the zeros of Riemann's $\zeta(s)$ but it is easy to see that the above argument of our proof can be applied to the single $\zeta(s)$ (i.e. $Q=1$) and hence (7.2) can be proved without referring to the density of the zeros.

Therefore without using the density argument we have obtained a a proof of

Theorem.

$$\sum_{q\le Q}\max_{(q,l)=1}\left|\psi(x+h;q,l)-\psi(x;q,l)-\frac{h}{\varphi(q)}\right|\ll h(\log x)^{-A}$$

with an arbitrarily large A and

$$x\ge h\ge x^{1-\theta},\qquad Q=x^{3/26(1-(8/3)\theta-\varepsilon)}$$

where ϑ is restricted by $(3/8)-10\varepsilon\ge\vartheta\ge 0$.

Concluding remark. As an easy application we have

$$\sum_{x\le p\le x+h}d(p-1)\ge c(\varepsilon)h,$$

where p denotes prime number and h satisfies

$$x\ge h\ge x^{5/8+\varepsilon}$$

and $c(\varepsilon)$ tends to zero as $\varepsilon\to+0$.

References

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