194. The Completion of Topological Spaces

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For a set X, the family consisting of all the filters in X is denoted by $F(X)$. For a topological space X, the collection of all the open sets of X is called *a topology* of X and denoted by $O(X)$.

Let X be a topological space. If a filter $\mathfrak{f}$ in X is generated by the filter base $\mathfrak{f} \cap O(X)$, then $\mathfrak{f}$ is an *open filter* in X. And the family consisting of all the open filters in X is denoted by $OF(X)$. Specially, for a point x of X, the open filter generated by the filter base $\{V \mid x \in V \in O(X)\}$ is called a *neighborhood system* of x and denoted by $\mathfrak{N}(x)$.

If a topological space X contains its dense subspace Y, then X is said to be an *extension* of Y.

Let a topological space X be an extension of Y. Then, for a point x of X and its neighborhood system $\mathfrak{N}(x)$, $\{V \cap Y \mid V \in \mathfrak{N}(x)\}$ is a *trace* of x on Y. We get a mapping φ of X into $OF(Y)$ such that, for every $x \in X$, $\varphi(x)$ is the trace of x on Y. This φ is called a *trace system* of X on Y. And the restriction $\varphi \mid X \setminus Y$ of φ on $X \setminus Y$ is a *tracer* of X on Y.

If φ is a trace system of an extension X of a topological space Y on Y, then X is said to be *extended* from Y by a tracer $\varphi \mid X \setminus Y$.

The following is the fundamental theorem of the extension theory of topological spaces.

Theorem 1. *Let Y be a topological space, X be a set containing Y and φ be a mapping of $X \setminus Y$ into $OF(X)$. Then there exists a topology of X such that X is an extension of which the tracer on Y is φ.*

In this paper, instead of this Theorem 1, Theorem 2 will be proved.

Example 1. Let X be the discrete topological space consisting of all the natural numbers, X^* be $X \cup \{\omega_1, \omega_2\}$, $\mathfrak{N}(\omega_1)$ be $\{A \cup \{\omega_1\} \mid A \subseteq X, X \setminus A \text{ is finite}\}$ and $\mathfrak{N}(\omega_2)$ be $\{A \cup \{\omega_2\} \mid A \subseteq X, X \setminus A \text{ is finite}\}$. Then X^* is a T_1 extension of X.

Example 2. Let X be the same as Example 1, X^* be $X \cup \{\omega_1, \omega_2\}$, $\mathfrak{N}(\omega_1)$ be $\{A \cup \{\omega_1, \omega_2\} \mid A \subseteq X, X \setminus A \text{ is finite}\}$ and $\mathfrak{N}(\omega_2) = \mathfrak{N}(\omega_1)$. Then X^* is an extension of X.

Example 3. Let R be the topological space of all the real numbers and S be the subspace of R consisting of all the rational numbers. Denote a trace of a real number x on S by $\varphi(x)$. If x is a rational number, put $\mathfrak{N}(x)=\varphi(x)$ and, for every irrational number x, put $\mathfrak{N}(x)=\{V\cup\{x\}\mid V\in\varphi(x)\}$. Then we get a topology of R such that $\mathfrak{N}(x)$ is the neighborhood system of x. This different topology of R is the strongest one in such topologies of R that R is a T_1 extension of which the tracer is $\varphi\mid R\backslash S$.

These three examples show the following.

Proposition. *Let X be a set containing a topological space Y and φ be a mapping of $X\backslash Y$ into $OF(X)$. Then a topology of X such that X is an extension of Y of which a tracer is φ is generally not unique.*

Let X^* be the completion of a uniform space X and E be the family consisting of all the non-convergent minimal Cauchy filters in X. Then X^* is an extension of X, $X^*=X\cup E$ and the tracer is the identity mapping of E. Furthermore filter $\mathfrak{f}$ in X^* is a Cauchy filter if and only if $\mathfrak{f}$ converges to some point of X^* in X^*. In other words, the topology of X^* determines the Cauchy filters in X^*.

Generally an extension of a topological space Y is not uniquely determined by Y and a tracer. But an extension of Y that satisfies some condition is uniquely determined by Y and its tracer. The completion theory[1] of topological spaces is to find such a condition of the topology.

Let X^* be an extension of a topological space X and φ be the tracer of X^* on X. If X^* has the weakest topology in such topologies of X^* that X^* is an extension of X of which a tracer on X is φ and every subset consisting of only one point of $X^*\backslash X$ is a closed set of X^*, then X^* is an *RT_1 extension* of X.

Theorem 2. *Let X^* be a set containing a topological space X and φ be a mapping of $X^*\backslash X$ into $OF(X)$. Then there exists a topology of X^* such that X^* is an RT_1 extension of X of which the tracer is φ.*

Proof. Let N be the family consisting of all the finite subsets of $X^*\backslash X$.

For every open set V of X, put

$$V'=\{x\mid x\in X^*\backslash X,\ V\in\varphi(x)\}$$
$$V^*=V\cup V'$$
$$\mathfrak{B}=\{V^*\backslash A\mid V\in O(X),\ A\in N\}.$$

$\mathfrak{B}$ generates a topology of the set X^*. By this topology, X^* becomes an extension of X such that φ is a tracer of X^* on X and any subset consisting of only one point of $X^*\backslash X$ is closed in X^*.

But, if a topology O of X^* satisfies that X^* is an extension of X, φ is its tracer on X and every subset consisting of only one point of $X^*\backslash X$ is closed in X^*, then O contains this family $\mathfrak{B}$.

Thus, by the topology of X^* generated by $\mathfrak{B}$, X^* becomes an RT_1

[1] More precisely Moore-Smith type completion theory.

extension of X of which the tracer is φ.

Theorem 3. *An RT_1 extension of a topological space X is uniquely determined by X and its tracer on X.*

Proof. Let X^* be an RT_1 extension of X, φ be its tracer on X and N be the family consisting of all the finite subsets of $X^*\backslash X$. For every open set V of X, put

$$V^*=\{x \mid \varphi(x) \ni V,\ x \in X^*\backslash X\} \cup V$$
$$\mathfrak{B}=\{V^*\backslash A \mid V \in O(X),\ A \in N\}.$$

$O(X^*)$ contains $\mathfrak{B}$. But $\mathfrak{B}$ generates a topology of X^* such that every subset consisting of only one point of $X^*\backslash X$ is closed and the tracer on X is φ.

So $O(X^*)$ is generated by $\mathfrak{B}$.

Thus $O(X^*)$ is uniquely determined by X and φ.

Example 4. The RT_1 extension of topological spaces is a completion of topological spaces.

References

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