190. A Note on Ribbon 2-Knots

By Akio ŌMAE
Department of Mathematics, Kōbe University

(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1971)

1. We shall consider the 2-spheres in a 4-sphere that are locally flat, which will be called 2-*knots*. S. Kinoshita [2] showed that for each polynomial $f(t)$ with $f(1)=\pm 1$, there exists a 2-sphere in a 4-sphere whose Alexander polynomial is defined and equal to $f(t)$. Recently, by an another method, D. W. Sumners [4] [5] showed that the existence of the 2-knot K^2 such that i) the Alexander polynomial of K^2 is $f(t)$ above, and moreover, ii) the second homotopy group of the complement of K^2 has the "Γ-torsion".

It is easy to see that the 2-knots which S. Kinoshita constructed in [2] are ribbon 2-knots [6] [7]. He gave us the following question.

"Is every Sumners's 2-knot a ribbon 2-knot?"

In this paper we will give the affirmative answer of this question. We will consider everything from the combinatorial standpoint of view. By S^n, $\mathring{X}$, ∂X and $N(X, Y)$, we shall denote an n-sphere, the interior of X, the boundary of X and the regular neighborhood of X in Y, respectively. $X \simeq Y$ means that X is homeomorphic to Y, and $\overset{m}{\#} X$ the connected sum of the m copies of X.

2. We will give some knowledge of ribbon and Sumners's 2-knots [5] [7].

Definition 2.1. A locally flat 2-sphere K^2 in S^4 will be called a *ribbon* 2-knot, if there is a ribbon map ρ of a 3-ball B^3 into S^4 satisfying the following conditions

(1) $\rho \mid \partial B^3$ is an embedding and $\rho(\partial B^3)=K^2$,

(2) the self-intersections of B^3 by ρ consists of mutually disjoint 2-balls $D_1^2, \cdots, D_s^2$,

(3) the inverse set $\rho^{-1}(D_i^2)$ consists of disjoint 2-balls $D_i'^2$ and $D_i''^2$ such that $D_i'^2 \subset \mathring{B}^3$ and $\partial D_i''^2 = D_i''^2 \cap \partial B^3$ $(i=1, \cdots, s)$.

Let N_i^3 be a spherical-shell, which is homeomorphic to $S^2 \times [0, 1]$ $(i=1, \cdots, m)$. A system of spherical-shells $N_1^3 \cup \cdots \cup N_m^3$ will be called *trivial* if they are mutually disjoint and such that

i) the 2-link $\partial N_1^3 \cup \cdots \cup \partial N_m^3$ of $2m$ components is of trivial type in $S^4 - (\mathring{N}_1^3 \cup \cdots \cup \mathring{N}_m^3)$; that is, there are mutually disjoint 3-balls $B_1^3, \cdots, B_{2m}^3$ in $S^4 - (\mathring{N}_1^3 \cup \cdots \cup \mathring{N}_m^3)$ such that $\partial N_i^3 = \partial B_i^3 \cup \partial B_{m+i}^3$ $(i=1, \cdots, m)$,

ii) for each i the 3-sphere $B_i^3 \cup N_i^3 \cup B_{m+i}^3$ bounds a 4-ball B_i^4 in S^4 such that $B_i^4 \cap B_j^4 = \emptyset$ $(i \neq j)$.

Let W^3 be a 3-manifold in S^4 which is homeomorphic to $\overset{m}{\#}(S^1\times S^2)-\mathring{\Delta}^3$, where Δ^3 is a 3-simplex. We will call W^3 in S^4 *semi-unknotted* if on it there is a trivial system of spherical-shells $N_1^3\cup\cdots\cup N_m^3$ which is such that $W^3-(\mathring{N}_1^3\cup\cdots\cup\mathring{N}_m^3)$ is homeomorphic to the closure of a 3-sphere removed of mutually disjoint $2m+1$ 3-balls [1]. From the theorem (3.6) in [7] we have

Lemma 2.2. *A 2-knot K^2 is a ribbon 2-knot, if and only if K^2 bounds a semi-unknotted 3-manifold W^3 in S^4.*

Construction of Sumners's 2-knot.

Let B^3 be a 3-ball in the boundary 4-sphere S^4 of a 5-ball B^5. Let $f: S^0\to S^4-B^3$ be an embedding, and attach a 1-handle h^1 to B^5 by f to obtain the manifold $T=B^5\cup_f h^1$. Let $S_0^2=\partial B^3$. Let α denote the generator of $\pi_1(\partial T-S_0^2)$ which goes around the handle and β the generator which links once S_0^2 in ∂T. Let $g: S^1\to\partial T-S_0^2$ be the embedding in the homotopy class of $\alpha^{a_0}\beta\alpha^{a_1}\beta\cdots\beta\alpha^{a_m}\beta^{-m}\in\pi_1(\partial T-S_0^2)$ such that $a_0+\cdots+a_m=\pm1$. Attaching a 2-handle h^2 to T by g, we obtain the manifold $T\cup_g h^2=(B^5\cup_f h^1)\cup_g h^2=\tilde{B}^5$ that is homeomorphic to a 5-ball from the handle cancellation theorem [3]. It is easy to see that S_0^2 is a 2-knot in the 4-sphere $\tilde{S}^4=\partial\tilde{B}^5$.

3. In this section, we will prove the following

Theorem 3.1. *Every Sumners's 2-knot is a ribbon 2-knot.*

Proof. It is sufficient to show that Sumners's 2-knot given in section 2 is a ribbon 2-knot.

The 3-ball B^3 and the attaching sphere $g(S^1)$ of 2-handle h^2 intersect at $2m$ points; say $x_1,\cdots,x_m,x_{-m},\cdots,x_{-1}$ whose order is according to the orientation of $g(S^1)$. Let $x_{-i}x_i$ be a subarc of $g(S^1)$ from x_{-i} to x_i in accordance with the orientation of $g(S^1)(i=1,\cdots,m)$. We may assume that $N(x_{-1}x_1,\partial T)$ and the 3-ball B^3 intersect at 3-balls D_{-1}^3 and D_1^3 whose centers are x_{-1} and x_1 respectively. Then D_{-1}^3 and D_1^3 divide $\partial N(x_{-1}x_1,\partial T)$ into two 3-balls and a spherical-shell $\tilde{N}_1^3$ which is homeomorphic to $S^2\times[0,1]$. Let $W_1^3=\{B^3-(D_{-1}^3\cup D_1^3)\}\cup\tilde{N}_1^3$, then $W_1^3\simeq S^1\times S^2-\mathring{\Delta}^3$, $W_1^3\cap g(S^1)=x_2\cup\cdots\cup x_m\cup x_{-m}\cup\cdots\cup x_{-2}$ and $\partial W_1^3=\partial B^3=S_0^2$. We can take a subdivision T_2 such that $N(x_{-2}x_2,\partial T_2)$ and W_1^3 intersect at 3-balls D_{-2}^3 and D_2^3 whose centers are x_{-2} and x_2 respectively. D_{-2}^3 and D_2^3 divide $\partial N(x_{-2}x_2,\partial T_2)$ into two 3-balls and a spherical-shell $\tilde{N}_2^3$. Let $W_2^3=\{W_1^3-(D_{-2}^3\cup D_2^3)\}\cup\tilde{N}_2^3$, then $W_2^3\simeq\overset{2}{\#}(S^1\times S^2)-\mathring{\Delta}^3$, $W_2^3\cap g(S^1)=x_3\cup\cdots\cup x_m\cup x_{-m}\cup\cdots\cup x_{-3}$ and $\partial W_2^3=\partial W_1^3=S_0^2$. Repeating of this procedure, we obtain the 3-manifold $W_m^3=W^3$ such that $W^3\simeq\overset{m}{\#}(S^1\times S^2)-\mathring{\Delta}^3$, $W^3\cap g(S^1)=\emptyset$ and $\partial W^3=S_0^2$, see Fig. 1.

It is easily seen that W^3 is in $\tilde{S}^4$. In fact, let T_{m+1} be a subdivision of T_m, then $N(g(S^1),\partial T_{m+1})$ is considered to be an attaching tube of

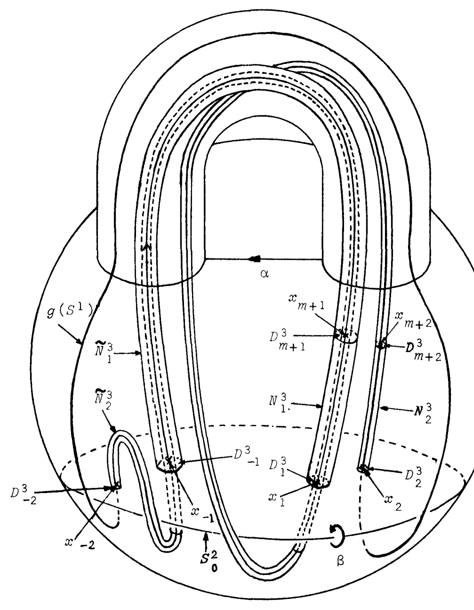

Fig. 1

2-handle h^2, and $W^3=W^3_m$ is in $\partial T-N(g(S^1),\partial T_{m+1})$, which is a subset of $\tilde{S}^4$.

We will show that W^3 is a semi-unknotted 3-manifold.

Let x_{m+i} be a point on the interior of $x_{i-1}x_i$ which is a subarc of $g(S^1)$ from x_{i-1} to x_i ($i=1,\cdots,m$) where x_0 means x_{-1}. Then there is a 3-ball D^3_{m+i} such that $x_{m+i}\in \mathring{D}^3_{m+i}\subset \mathring{N}(x_{-i}x_i,\partial T)$, $\partial D^3_{m+i}\subset \tilde{N}^3_i$ and ∂D^3_{m+i} divides $\tilde{N}^3_i$ into two spherical-shells. Let N^3_i ($i=1,\cdots,m$) be the one of the two spherical-shells with the boundary $\partial D^3_i\cup\partial D^3_{m+i}$. Let S^2_i and S^2_{m+i} be the 2-spheres ∂D^3_i and ∂D^3_{m+i}, respectively. Then the system of spherical-shells $N^3_1\cup\cdots\cup N^3_m$ will be trivial.

Since $g(S^1)$ is ambient isotopic to α in $\partial T=\alpha\times S^3$, it is considered that $\partial T=g(S^1)\times S^3$ and D^3_i is in $x_i\times S^3$ ($i=1,\cdots,2m$). Since $N(g(S^1),\partial T_{m+1})\cap(x_i\times S^3)=x_i\times$3-ball, S^2_i bounds a 3-ball $B^3_i=x_i\times S^3-\mathring{D}^3_i$ in $x_i\times S^3-(N(g(S^1),\partial T_{m+1})\cap(x_i\times S^3))-(\mathring{N}^3_1\cup\cdots\cup\mathring{N}^3_m)\subset\tilde{S}^4-(\mathring{N}^3_1\cup\cdots\cup\mathring{N}^3_m)$, see Fig. 2. Therefore $\partial N^3_1\cup\cdots\cup\partial N^3_m=S^2_1\cup\cdots\cup S^2_{2m}$ is a trivial link in $\tilde{S}^4-(\mathring{N}^3_1\cup\cdots\cup\mathring{N}^3_m)$. From the construction of N^3_i and B^3_j, we can easily see that each $B^3_i\cup N^3_i\cup B^3_{m+i}$ bounds the 4-ball $B^4_i=B^3_i\times x_ix_{m+i}$ in $\partial T-N(g(S^1),\partial T_{m+1})\subset\tilde{S}^4$ and $B^4_1,\cdots,B^4_m$ are mutually disjoint. Hence the system of spherical-shells $N^3_1\cup\cdots\cup N^3_m$ is trivial.

It is easy to see that $W^3-(\mathring{N}^3_1\cup\cdots\cup\mathring{N}^3_m)$ is homeomorphic to the closure of a 3-sphere removed of mutually disjoint $2m+1$ 3-balls. Hence W^3 is a semi-unknotted 3-manifold.

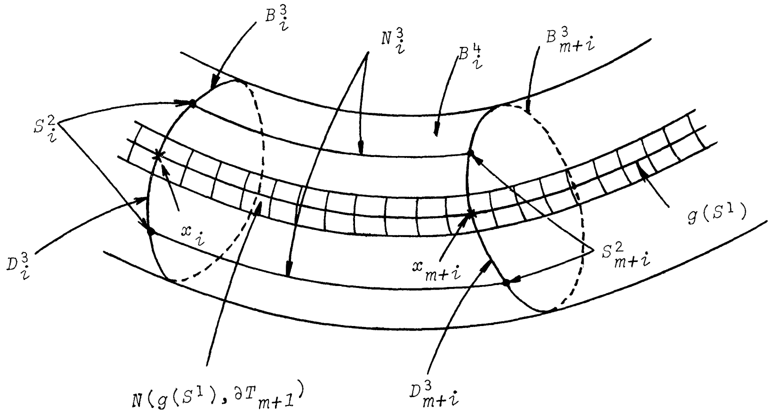

Fig. 2

Therefore, from Lemma 2.2, S_0^2 is a ribbon 2-knot. This completes the proof of Theorem 3.1.

References

[1] R. H. Fox and H. F. Trotter: Characterization of slices and ribbons. Osaka J. Math. (to appear).

[2] S. Kinoshita: On the Alexander polynomials of 2-spheres in a 4-sphere. Ann. of Math., **74**, 518–531 (1961).

[3] J. R. Stallings: Lectures on Polyhedral Topology. Tata Inst. of Fund. Res., Bombay (1967).

[4] D. W. Sumners: Higher-dimensional slice knots. Bull. Amer. Math. Soc., **72**, 894-897 (1966) .

[5] ——: Homotopy torsion in codimension two knots. Proc. Amer. Math. Soc., **24**, 229–240 (1970).

[6] T. Yajima: On simply knotted spheres in R^4. Osaka J. Math., **1**, 133–152 (1964).

[7] T. Yanagawa: On ribbon 2-knots. Osaka J. Math., **6**, 447-464 (1969).