185. δ_p and Countably Paracompact Spaces

By J. M. BOYTE
Appalachian State University

(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1971)

In [3], Mack defines the term δ-normal, and proves that if I is the closed unit interval, then a space X is countably paracompact if and only if $X\times I$ is δ-normal. In this paper we define the term δ_p which is stronger than δ-normal, but is strictly weaker than countable paracompactness, and is strictly weaker than normality; and we prove the following:

Theorem 1. *The following are equivalent for a space X*

(i) *The space X is countably paracompact.*

(ii) *The space X is δ_p and countably metacompact.*

(iii) *The space X is δ_p and every countable open cover of X has a countable semi-refinement of closed sets.*

(iv) *If C is a countable open cover of X, then there exists a countable collection $L=\{L_i\mid i=1,2,\cdots\}$ of open refinements of C such that for each $x\in X$ there is some L_i that is locally finite with respect to x.*

(v) *If I is the closed unit interval, then $X\times I$ is δ_p.*

We observe that (ii) of the above theorem is a slight generalization of a condition proven by Dowker [2]; further; we point out that Theorem 1 in [3] is used in proving (v) of the above theorem.

Definition. If X is a space and C is an open cover of X, then L is a semi-refinement of C if each member of L is contained in the union of a finite subset of C.

Definition. If X is a space and L is a collection of subsets of X, then L is locally finite with respect to a subset A of X, if for each $x\in A$, there exists an open set V, $x\in V$, such that V intersects only finitely many members of L.

Definition. Let X be a space and let N be a cardinal number. Then X is called an N_p space, if for each open cover C, cardinality of C less than or equal N, there exists for each closed set F contained in any member of C, an open refinement of C that is locally finite with respect to F. In the special case when $N=$aleph zero, we will denote N_p by δ_p.

For an infinite cardinal N, a topological space is N-normal if each pair of disjoint closed sets, one of which is a regular G_N-set, have disjoint neighborhoods [3]. A set B is called a regular G_N-set if it is the intersection of at most N closed sets whose interiors contain B [3].

For $N=$ aleph zero, N-normal is denoted by δ-normal and a regular G_N set by regular G_δ set.

A space X is called countably metacompact if every countable open cover of X has a point finite open refinement. The terminology of [4] is followed except that we shall use $V(x)$, $N(x)$, etc. (resp., $V(A)$, $N(A)$, etc.) to denote open sets containing the point x (resp., the subset A).

At the end of the paper, we give an example of a δ-normal countably metacompact space that is not countably paracompact, thus, by Theorem 1, a δ-normal space that is not δ_p.

Theorem 2. *If N is a cardinal number, then we have the following:*

(i) *Every paracompact space is N_p.*

(ii) *Each normal space is N_p.*

(iii) *Each N_p space is N-normal.*

Proof of Theorem 2. The proof of (i) is clear by the definition. To show (ii), let $C=\{G_\alpha \mid \alpha \in A\}$ be an open cover of a normal space X, and suppose a closed set $F_\lambda \subset G_\lambda$ for some $\lambda \in A$. Then there exists $V(F_\lambda)$ such that $\overline{V(F_\lambda)} \subset G_\lambda$. If we let $C_1=\{G_\lambda\} \cup \{(X-\overline{V(F_\lambda)}) \cap G_\alpha \mid \alpha \in A\}$, then C_1 is an open refinement of C that is locally finite with respect to F_λ. Thus, X is an N_p space for all cardinal N.

Suppose N is a cardinal number, and X is an N_p space. Assume F and K are disjoint closed sets with F a regular G_N-set. By definition, $F=\cap\{\overline{W_\alpha(F)} \mid \alpha \in A, \text{ card. } A \leq N\}$. Since $F \subset X-K$ and since $C=\{X-K\} \cup \{X-\overline{W_\alpha(F)} \mid \alpha \in A\}$ is an open cover of X, there exists an open refinement $\{V_\lambda \mid \lambda \in \Lambda\}$ of C that is locally finite with respect to F. For each $x \in F$, there exists $N(x)$ such that $N(x) \cap V_\lambda \neq \emptyset$ for only finitely many $\lambda \in \Lambda$. By letting $L_x=\{V_\lambda \mid V_\lambda \cap N(x) \neq \emptyset$ and $V_\lambda \subset X-\overline{W_\alpha(F)}$ for some $\alpha \in A\}$, there exists $H(x)$ such that $H(x) \cap (\cup L_x)=\emptyset$. Put $M(x)=N(x) \cap H(x)$; then $M(x) \cap (\cup\{V_\lambda \mid \lambda \in \Lambda$ and $V_\lambda \subset X-\overline{W_\alpha(F)}$ for some $\alpha \in A\})=\emptyset$. If we let $O(F)=\cup\{M(x) \mid x \in F\}$, then $O(F)$ is open and contains F. Further, if $V_\lambda \subset X-\overline{W_\alpha(F)}$, $\lambda \in \Lambda$ and $\alpha \in A$, then $O(F) \cap V_\lambda=\emptyset$. Hence $\overline{O(F)} \cap (\cup\{V_\lambda \mid \lambda \in \Lambda$ and $V_\lambda \subset X-\overline{W_\alpha(F)}\})=\emptyset$; and therefore, $\overline{O(F)} \subset X-K$. It follows that X is N-normal.

Definition. For each $i=1, \cdots, n$, let L_i be a collection of sets. Then $\bigcap_{i=1}^n L_i=\{\bigcap_{i=1}^n (V_{(i,\alpha)}) \mid V_{(i,\alpha)} \in L_i\}$.

Lemma 3. *Let X be a space, and let C be a countable open cover of X. If there exists a countable collection $L=\{L_i \mid i=1,2, \cdots\}$ of open refinements of C such that for each $x \in X$, there exists some L_i that is locally finite with respect to x, then there exists an open locally finite refinement of C.*

Proof. Let $C=\{G_i \mid i=1,2, \cdots\}$ be a countable open cover of X, and let $L=\{L_i \mid i=1,2, \cdots\}$ be a countable collection of open refinements of C such that for each $x \in X$ there exists some L_i that is locally finite with respect to x. For each positive integer n, let $L^n = \{V \mid V \in \bigcap_{i=1}^n L_i, V \not\subset \bigcup_{i=1}^n (G_i)\}$; and let $A_n = \{x \mid x \in X$ and there exists $M(x)$, $M(x) \cap (\cup L^n) = \emptyset\}$. To show that $\bigcup_{i=1}^\infty (A_i) = X$, let $x \in X$. Then there exists $N(x)$ and there exists a positive integer k such that $N(x)$ intersects only finitely many members of L_k; hence, there exists a positive integer K such that $V \in L_k$, $N(x) \cap V \neq \emptyset$ implies $V \subset \bigcup_{i=1}^K (G_i)$. Thus, $N(x) \cap (\cup L^K) = \emptyset$, and therefore, $x \in A_K$. If n is a positive integer, there exists $W(A_n)$ such that $\overline{W(A_n)} \subset \bigcup_{i=1}^n (G_i)$. Now $H = \{G_1\} \cup \{G_n - \bigcup_{i=1}^{n-1} (\overline{W(A_i)}) \mid n = 1, 2, \cdots\}$ is an open locally finite refinement of C.

Proof of Theorem 1. The proof that (i) implies (ii) is clear, (ii) implies (iii) is not very difficult; and (iv) implies (i) follows from Lemma 3.

To see that (iii) implies (iv), let $C = \{G_i \mid i = 1, 2, \cdots\}$ be a countable open cover of X, and let $\{F_j \mid j = 1, 2, \cdots\}$ be a countable closed semi-refinement of C. Then for each positive integer j, there exists a finite union $M_j = \bigcup_{k=1}^n (G_{(k,j)})$ of subsets of C such that $F_j \subset M_j$. If we let $C_j = \{M_j\} \cup C$, then there exists an open refinement H of C_j that is locally finite with respect to F_j. By letting $L_j = \{V \mid V \in H$ and $V \subset G_i$ for some $G_i \in C\} \cup \{V \cap G_{(k,j)} \mid k = 1, \cdots, n, M_j = \bigcup_{k=1}^n G_{(k,j)}, V \subset M_j$ and $V \in H\}$, L_j is an open refinement of C that is locally finite with respect to F_i. Therefore, a desired collection is $L = \{L_j \mid j = 1, 2, \cdots\}$.

That (i) implies (v) follows from Theorem 1 [2], and (v) implies (i) follows from Theorem 2 and Theorem 1 [3].

Comment. That every regular Lindelof space is paracompact follows from (iv) of Theorem 1.

A space X is defined to be point (countably) paracompact if for each (countable) open cover C of X, there exists for each $x \in X$ an open refinement of C that is locally finite with respect to x.

Remark. It is clear that each T_1 δ_p space is point countably paracompact, and that if X is a T_1 space that is an N_p space for each cardinal N, then X is point paracompact. By Theorem 2 [1], each T_2 point paracompact space is regular. Thus, if X is a T_2 space that is an N_p space for each infinite cardinal N, then X is regular, and the following proposition is obtained.

Proposition 4. *A T_2 space is normal if and only if it is an N_p space for every infinite cardinal N.*

Corollary 5. *Each T_2 paracompact space is normal.*

Example. There exists a δ-normal countably metacompact space that is not countably paracompact.

Let $X = \{(1/n, 0) \mid n = 1, 2, \cdots\} \cup \{(1, 0)\} \cup \{(1/n, 1) \mid n = 1, 2, \cdots\}$. The basic open sets are $\{(1/n, 1), (1/n, 0)\}$, $\{(1/n, 0)\}$, and $\{(1/n, 0) \mid n = 1, 2,$

$\cdots\} \cup \{(1, 0)\}$. It is easy to see that the space is countably metacompact but not countably paracompact. The only regular G_δ subsets of X are X and the empty set. Therefore x is δ-normal.

References

[1] J. M. Boyte: Point (countable) paracompactness. J. Australian Math. Soc. (to appear).

[2] C. H. Dowker: On countably paracompact spaces. Canad. J. Math., **3**, 219–224 (1951).

[3] J. Mack: On countable paracompactness and weak normality properties. Trans. Amer. Math. Soc., **148**, 265–271 (1970).

[4] J. Dugundji: Topology. Boston (1965).