184. Unimodular Numerical Contractions in Hilbert Space

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1. Let T be a unitary operator on a Hilbert space H. Then in particular

(i) T is a numerical contraction, i.e., $w(T) \leq 1$

(ii) the spectrum of T is a subset of the unit circle, i.e., $\sigma(T) \subset \{z, |z|=1\}$.

Call an arbitrary operator T a unimodular numerical contraction (u.n.c.) if it satisfies conditions (i) and (ii) above. Then similars to problems considered by B. Russo for unimodular contractions come to mind. The present paper has the aim to give some results in this direction.

2. In this section we give some easy results on unimodular numerical contractions concerning the eigenvalues.

Theorem 2.1. *If T is a unimodular numerical contration, then the following assertions hold:*

(i) *the eigenvectors of T corresponding to distinct eigenvalues are orthogonal,*

(ii) *if the eigenvectors of T span H, then T is unitary.*

Proof. For every $\xi \in \sigma(T)$ let $E_\xi(T) = \{x, Tx = \xi x\}$ and thus if T is a unimodular numerical contraction then

$$\xi \in \sigma(T) \cap \partial W(T)$$

and by Theorem 2 in [4], ξ is a normal eigenvalue, i.e.,

$$E_\xi(T) = E_{\xi*}(T^*).$$

From this (i) follows immediately.

The assertion (ii) follows from (i) as in the case of unimodular contractions.

For the following result we need the notion of a maximal-single-valued extension of $R(z, x) = (T-z)^{-1}x$, $z \in \rho(T) = C_{\sigma(T)}$ and the following

Difinition 2.1. If T is a unimodular numerical contraction such that there does not exist an invariant subspace of T on which T is normal we call T completely unimodular numerical contraction.

Theorem 2.2. *Every completely unimodular numerical contraction has the property of maximal single-valued extension.*

Proof. If no, we find a number $\xi \in \sigma(T)$ and a non zero element $x \in H$ such that $Tx = \xi x$ and thus

$$E_\xi(T) \neq \{0\}$$

and $T|_{E_\xi(T)}$ is normal; a contradiction.

3. In this section we give some results concerning sufficient conditions implying normality of numerical unimodular contractions and strong asymptotic behavior.

Theorem 3.1. *If T is a (u.n.c.) and for some integer n, T^n is normal then T is unitary.*

Proof. Let $w(T)$ be the numerical radius of T. Since $T^n = N$ is normal,

$$T^{*n}T = TT^{*n}$$

and so that we can apply the result in [5] about multiplicative character of numerical radius for operators,

$$w(T^{-n+1}) = w(T^{-n}T) \leq w(T^{-n})w(T) \leq 1$$

which gives that T^{-n+1} is normal and also T^{n-1} is normal. Continuing in this way we obtain that T is normal (unitary).

Now we give a result on invariant subspaces suggested by the proof of above theorem.

The following result was stated by L. de Branges (Math. Rev., Vol. 26, No. 1759 (1963)) and proved by K. Kitano.

Theorem 6. *If $T^*T - I$ is an operator of the class C_p for some $p \geq 1$ then T has a proper invariant subspace.*

Our result is

Theorem 3.2. *If for some integers m, n, p, the essentially invertible operator T has the property*

1) $T^m T^{*n} - I$

2) $T^p T^{*p} - I$

are compact operators, then T has a proper invariant subspace $(m \neq n)$.

Proof. Recall that 'essentially invertible' means invertibility modulo compact operators, i.e., in the Calkin algebra.

Since $\hat{T}$ is invertible, from 2) it follows that

$$\hat{T}^p\hat{T}^{*p} - \hat{I} = 0$$

and by a result of Gindler and Taylor [3] $\hat{T}^p$ is unitary so that $\hat{T}$ is by Theorem 3.1 unitary. If $m > n$ we have that

$$T^{m-n} - I = \text{compact}$$

and the theorem is proved.

In the next theorem we give a structure theorem for polynomially compact (u.n.c.) operator.

Theorem 3.3. *If for some polynomial $p(\lambda) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$ $p(T)$ is the zero operator and if T is (u.n.c.), then T is unitary.*

Proof. For every λ_i, let

$$M_i = \{x, \ Tx = \lambda_i x\}$$

be reducing and orthogonal (also different from $\{0\}$). If $\mathcal{M} = \oplus_i M_i$ then $T|_{\mathcal{M}}$ is unitary and $\sigma(T) = \sigma(T|_{\mathcal{M}}) \cup \sigma(T|_{\mathcal{M}^\perp})$. If $T|_{\mathcal{M}^\perp}$ is different from $\{0\}$ then we obtain easily a contradiction.

We conjecture that the theorem is true under more general conditions, i.e., if for some polynomial $p(\lambda)$, $p(T)$ is compact or $p(T)$ has the real part compact (or the imaginary part compact). However we are unable to prove this.

Using a result of F.Gilfeather [2] for a contraction operator on a Hilbert space H, we obtain a similar result for a (u.n.c.) operator T, which gives a necessary and sufficient spectral condition to be strong asymptotically convergent, that is, $\{T^n\}$ converges in the strong operator topology.

Theorem 3.4. *Let T be a (u.n.c.) operator for which $\sigma(T)$ is countable. Then T is strong asymptotically convergent if and only if $\sigma_p(T) \subset \{1\}$.*

Proof. It is known that a (u.n.c.) operator on H is similar to a contraction [8] (i.e., there exists a bounded linear operator S with a bounded inverse in H and an operator T_1 with $\|T_1\| \leq 1$ such that $T = S^{-1}T_1 S$). Then we have that $T^n = S^{-1}T_1^n S$.

From the similarity property, T_1 has the same spectrum and the same point spectrum and therefore T_1 satisfies the hypothesis of Proposition 2 [2]. It follows that T_1 is strong asymptotically convergent and then T is strong asymptotically convergent.

As a corollary we obtain that: if T is a polynomially compact (u.n.c.) operator on H then T is strong asymptotically convergent if and only if $\sigma_p(T) \subset \{1\}$.

References

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