231. A Remark on the Boundary Behavior of $(Q)L_1$-Principal Functions

By Mineko WATANABE
Yoshida College, Kyoto University

(Comm. by Kinjirô KUNUGI, M. J. A., Dec. 13, 1971)

Let R be an open Riemann surface and Q be the canonical partition of the ideal boundary of R. The problem characterizing $(Q)L_1$-principal functions by the boundary behavior under compactifications has been investigated by several authors (Sario-Oikawa [9]). The class of $(Q)L_1$-principal functions has been shown to be identical with the class of single-valued canonical potentials introduced by Kusunoki [5] (Watanabe [10]). As a necessary condition, the fact that a $(Q)L_1$-principal function can be extended almost everywhere (or quasi-everywhere) continuously on some compactifications so that the extension is a.e. (q.e.) constant on each component of the ideal boundary has been proved by some authors in different ways (Ikegami [3], Kusunoki [6] and Watanabe [10]).

Then, the question arises whether, conversely, this boundary property would be sufficient for a function to be a $(Q)L_1$-principal function.

Watanabe [10] showed a sufficient condition in the following particular form. Suppose that a real-valued harmonic function f with a finite number of singularities is Dirichlet integrable in a boundary neighborhood U and $\int_\gamma {}^*df=0$ for any dividing cycle γ in U, and is almost everywhere constant on each boundary component of a compactification R^*. The R^* may be one of Martin, Royden, Wiener, Kuramochi or a $\mathfrak{Q}$-compactification denoting by $\mathfrak{Q}$ a sublattice of HP which contains constant. If the set of constant values taken by f on boundary components is isolated except the supremum and infimum, then f is a $(Q)L_1$-principal function.

On the other hand, if R is of finite genus, any harmonic function in a boundary neighborhood whose conjugate is semi-exact has a limit at a weak boundary component. Therefore, if a Riemann surface, whose all boundary components are weak, is not of class O_{KD}, there exist functions which are not $(Q)L_1$-principal functions but have limits at any boundary component (Watanabe [10]). However, these functions do not seem to be good enough as counter examples, because the condition 'having limits at weak boundary components' may not be expected to be any restriction.

We are now going to show the following

Theorem. *There exists a Riemann surface carrying boundary components of positive capacity, and on which there exists a function f real harmonic except a finite number of singularities and satisfying the following conditions:*

i) *f is Dirichlet integrable on a boundary neighborhood U and $\int_\gamma {}^*df=0$ for any dividing cycle γ contained in U,*

ii) *f can be extended continuously to the Kerékjártó-Stoilow compactification**), *and*

iii) *f is not a $(Q)L_1$-principal function on R.*

In short, the conditions i) and ii) are not sufficient for f to be a $(Q)L_1$-principal function without further restrictions.

The essential idea to construct such a function is the following. Let R^* be a compactification of type S of R. Suppose that the boundary $\Delta=R^*-R$ consists of two parts Δ_1 and Δ_2, where all components of Δ_1 are weak and all components of Δ_2 are not semi-weak, and there are neighborhoods U_1 of Δ_1 and U_2 of Δ_2 such that $\bar{U}_1\cap\bar{U}_2=\phi$. As a normal operator L defined with respect to the boundary neighborhood $U=U_1\cup U_2$, we take $L=L_0$ in U_1 and $L=(Q)L_1$ in U_2. If the number of components of Δ_1 is sufficiently large, the function on R constructed by the operator L is different from $(Q)L_1$-principal functions.

For a finite number of given singularities s with vanishing flux on R, and a canonical region Ω carrying all the s, we construct the L_0-principal function $f_{0\Omega}$ and the $(Q)L_1$-principal function $f_{1\Omega}$ on Ω with the singularities s as follows. The normal derivative of the $f_{0\Omega}$ vanishes on the boundary $\partial\Omega$ of Ω, and the $f_{1\Omega}$ is constant on each component of $\partial\Omega$ and the flux of $f_{1\Omega}$ vanishes over each component of $\partial\Omega$. Then, the suitably normalized families $\{f_{i\Omega}\}_\Omega$ $(i=0,1)$ converge almost uniformly to f_i $(i=0,1)$ on R, where f_0 is the L_0-principal function and f_1 is the $(Q)L_1$-principal function on R with the singularities s (Rodin-Sario [8]). Moreover, $\|df_{i\Omega}-df_i\|_\Omega$ $(i=0,1)$ converge to zero when Ω tends to R (Watanabe [10]). The operator L defined above is also normal and we can easily show that these two converging properties hold good for a function constructed by the operator L.

In order to prove the Theorem, we practically construct a Riemann surface and a function on it as follows. Let $\tilde{R}$ be a Riemann surface of genus zero and whose all boundary components are weak. Assume that $\tilde{R}$ is not of class O_{KD}. Let $\tilde{R}^*$ be a compactification of $\tilde{R}$. Then $\tilde{R}^*$ is a closed Riemann surface of the same genus as $\tilde{R}$ and it is

*) This is clearly equivalent to the following statement: f can be extended continuously to any compactification of type S in the sense of Constantinescu-Cornea [2] so that the extension is constant on each boundary component.

topologically unique (Jurchescu [4]). For a finite number of singularities s on $\tilde{R}$, we construct an L_0-principal function $\tilde{f}_0$ on $\tilde{R}$ with the s. The niveau curves of $\tilde{f}_0$ are analytic except isolated singular points. Along some niveau curves of $\tilde{f}_0$, we remove positive length of non-closed curves from $\tilde{R}$ outside of a boundary neighborhood U_1 of $\tilde{R}$ and we denote the removed set by Δ_2. The remaining part R is a Riemann surface of the same genus as $\tilde{R}$, and the $\tilde{R}^*$ is also a compactification of type S of R. The boundary $\Delta=\tilde{R}^*-R$ of R consists of two parts $\Delta_1=\tilde{R}^*-\tilde{R}$ and $\Delta_2=\tilde{R}-R$. All components of Δ_1 are weak, because the weakness of a boundary component is a γ-property (Jurchescu [4]), and all components of Δ_2 are not semi-weak. We choose a neighbourhood U_2 of Δ_2 so that $\bar{U}_1\cap\bar{U}_2=\phi$. The restriction f of $\tilde{f}_0$ to R is the function constructed by the operator L, and the extension of f to $\tilde{R}^*$ is constant on each boundary component of R. We take the $(Q)L_1$-principal function f_1 on R with the same singularities s. Then f_1 can be extended continuously to $\tilde{R}^*$ so that the extension is constant on each boundary component of R (Watanabe [10]).

The final step to reach our conclusion is to show that the $f-f_1$ is not constant on R. Assume that $f-f_1$ is constant on R. Then f_1 can be extended harmonically to $\tilde{R}$ and the extension $\tilde{f}_1$ is a $(Q)L_1$-principal function on $\tilde{R}$. Moreover $\tilde{f}_0-\tilde{f}_1$ is constant on $\tilde{R}$. But this is a contradiction, because $\tilde{R}$ is not of class O_{KD} and an L_0-principal function and a $(Q)L_1$-principal function with the same singularities coincide each other if and only if $\tilde{R}\in O_{KD}$ (Ahlfors-Sario [1]).

Another example is a planar Riemann surface which has no weak boundary components. Let $\boldsymbol{C}$ be the extended complex plane and E be the following set in $\boldsymbol{C}$.

$$E=\{z=x+iy \mid x\in A, 0\leq y\leq 1\},$$

where A is a generalized Cantor set of positive linear measure in [0, 1]. Then E has positive planar Lebesgue measure. Let R be $\boldsymbol{C}-E$, then E is the boundary of R, and it is readily seen that any component of the E is not a weak boundary component. Further, for any compactification R^* of type S, a boundary component of R on R^* corresponds to a component of the E and vice versa. The function $f(z)=\mathrm{Re}\, z=x$ is real harmonic with the only singularity at the point at infinity, and Dirichlet integrable on a boundary neighborhood. Moreover, f is constant on each component of E. We construct the $(Q)L_1$-principal function f_1 on R with the singularity Re z at the point at infinity. Because the mapping $h=f_1+if_1^*$ of R is one to one and the complement of the image of R by h is of Lebesgue measure zero (Ahlfors-Sario [1]), we know that f_1-f is not constant on R, or f is not a $(Q)L_1$-principal function.

As for regular harmonic functions, we already know that an integral of any differential of class Γ_{hm} can be extended a.e. (q.e.) continuously to some compactifications of type S so that the extension is a.e. (q.e.) constant on each boundary component (Kusunoki [7] and Watanabe [10]). Let us denote by Γ_{hQ} the subclass of Γ_{he} which consists of those differentials whose integrals have the boundary property just stated. It is evident that *the conditions* i) *and* ii) *in the Theorem characterize* $(Q)L_1$*-principal functions if and only if the* Γ_{hQ} *coincides with* Γ_{hm} *on a Riemann surface.* For, we have

$$d(f-f_1)\in\Gamma_{hQ}\cap\Gamma^*_{hse},$$

where f is a function with the properties i) and ii), and f_1 is a $(Q)L_1$-principal function with the same singularities as f, and we have the orthogonal decomposition

$$\Gamma_{hQ}=\Gamma_{hm}\oplus\Gamma_{hQ}\cap\Gamma^*_{hse}.$$

If a Riemann surface is of class O_{KD}, or if a number of boundary components of a Riemann surface is finite, it holds that $\Gamma_{hQ}=\Gamma_{hm}$ (cf. Theorem 2 in Watanabe [10]).

By observing the function $u=f_1-f$ in the above examples, we obtain the following

Corollary. *There is a Riemann surface on which there exists a function u such that*

i) *the continuous extension of u to a compactification of type S is constant on each boundary component, and*

ii) *du is an element of class* $\Gamma_{he}\cap\Gamma^*_{hse}$, *or not of class* Γ_{hm}.

References

[1] Ahlfors, L. V., and Sario, L.: Riemann Surfaces. Princeton (1960).

[2] Constantinescu, C., und Cornea, A.: Idealeränder Riemannscher Flächen. Springer-Verlag (1963).

[3] Ikegami, T.: A mapping of Martin boundary into Kuramochi boundary by means of poles. Osaka J. Math., **5**, 285–297 (1968).

[4] Jurchescu, M.: Modulus of a boundary component. Pacific J. Math., **8**, 791–809 (1958).

[5] Kusunoki, Y.: Theory of Abelian integrals and its applications to conformal mappings. Mem. Coll. Sci. Univ. Kyoto, Ser. A, **32**, 235–258 (1959).

[6] ——: Characterizations of canonical differentials. J. Math. Kyoto Univ., **5**, 197–207 (1966).

[7] ——: On some boundary properties of harmonic Dirichlet functions. Proc. Japan Acad., **46**, 277–282 (1970).

[8] Rodin, B., and Sario, L.: Convergence of normal operators. Kôdai Math. Sem. Rep., **19**, 165–173 (1967).

[9] Sario, L., and Oikawa, K.: Capacity Functions. Springer-Verlag (1969).

[10] Watanabe, M.: On a boundary property of principal functions. Pacific J. Math., **31**, 537–545 (1969).