224. Results Related to Closed Images of M-Spaces. I

By Kiiti MORITA*) and Thomas RISHEL**)

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1. Introduction. In 1969, J. Nagata began a discussion of characterizations of images of M-spaces under various continuous maps (see, for instance [4] and [5]). Other images of M-spaces have been characterized by Wicke [7], Chiba [1] and by Rishel [6]. One such characterization which has not yet been carried out is that of closed images of M-spaces. It is the purpose of this paper to demonstrate that characterization.

In this paper, all maps are continuous and onto; the symbol "N" will refer to the natural numbers. All spaces will be considered to be T_1-spaces.

2. Preliminaries about covers.

Definition 2.1. A system $\{F_\alpha : \alpha \in \Omega\}$ of closed sets from a space X is said to be *hereditarily closure preserving* if and only if: for any system $\{M_\alpha : \alpha \in \Omega\}$ of closed sets in X such that $M_\alpha \subset F_\alpha$ for every $\alpha \in \Omega$, $\cup\{M_\alpha : \alpha \in \Omega\} = \mathrm{Cl}\,[\cup\{M_\alpha : \alpha \in \Omega\}]$.

Definition 2.2. A family $\{B_n : n \in N\}$ of sets in a space X is said to form a *q-sequence at* $x \in X$ if and only if:

(a) $x \in B_n$ for every $n \in N$,

(b) for every point-sequence $\{x_n\}$ such that $x_n \in B_n$ for every $n \in N$, $\{x_n\}$ clusters.

Definition 2.3. A sequence of closed covers $\{\mathcal{A}_n\}$ of a space X is said to be *almost q-refining* if and only if for any point $x \in X$, any system of sets $\{B_n\}$, such that $B_n \in \mathcal{A}_n$ for all $n \in N$ and $x \in B_n$ for all $n \in N$, is either hereditarily closure preserving or else forms a q-sequence at x.

Morita [3] originally defined M-spaces.

Definition 2.4. A space X is said to be an *M-space* if and only if there exists a normal sequence of open covers $\{\mathcal{U}_1, \mathcal{U}_2, \cdots\}$ of X satisfying

(1) every point-sequence of the form $\{x_n\}$, where $x_n \in \mathrm{St}(x, \mathcal{U}_n)$ for all n and for fixed $x \in X$, has a cluster point.

Definition 2.5 (Nagata [5]). A space Y is *quasi-k* if and only if, given $F \subset Y$, F is closed whenever $F \cap K$ is relatively closed in K for every

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*) Tokyo University of Education.

**) Dalhousie University. This author's work was begun at Dalhousie University and completed at Tokyo University of Education under Canadian Research Council Grant A-3999.

countably compact $K \subset Y$.

3. Characterization. We now prove our main result.

Theorem 3.1. *Let Y be a regular space. Y is the closed image of a regular M-space X if and only if Y has the following conditions:*

(a) *Y is quasi-k;*

(b) *there exists in Y an almost q-refining sequence $\{\mathcal{F}_n\}$ of hereditarily closure preserving closed covers such that every point $y \in Y$ has a q-sequence $\{A_n\}$ with $A_n \in \mathcal{F}_n$ for every $n \in N$.*

(i) **Proof of the "if" part.**

Let $\mathcal{F}_i = \{F_{i\alpha} : \alpha \in \Omega_i\}$, $i \in N$. Let $B = \{\alpha = (\alpha_1, \alpha_2, \cdots) : \{F_{i\alpha_i}\}$ is a q-sequence at some $y \in Y$, and $F_{i\alpha_i} \in \mathcal{F}_i$ for all $i \in N$.$\}$. Topologize B as a subspace of a Baire metric space. A neighbourhood of $\alpha = (\alpha_1, \alpha_2, \cdots) \in B$ will have the form

$$B(\alpha_1, \cdots, \alpha_n) = \{\beta \in B : \beta_i = \alpha_i \text{ for all } i \text{ with } 1 \leq i \leq n\}.$$

Take a subspace X of $B \times Y$ as follows:

$$X = \left\{(\alpha, y) : \alpha = (\alpha_1, \alpha_2, \cdots) \in B, y \in \bigcap_{i=1}^{\infty} F_{i\alpha_i}\right\}.$$

Since Y is regular, so is X.

Define a map $\varphi : X \to B$ by

$$\varphi(\alpha, y) = \alpha, \quad \text{where} \quad y \in \bigcap_{i=1}^{\infty} F_{i\alpha_i}.$$

Let C be any subset of X, and let $\alpha \in \mathrm{Cl}\, \varphi(C)$.

Let $(\beta^{(n)}, y_n) \in C \cap B(\alpha_1, \cdots, \alpha_n) \times Y$, $n \in N$. Since $y_n \in F_{n\alpha_n}$, $n \in N$, and $\{F_{i\alpha_i}\}$ is a q-sequence, $\{y_n\}$ has a cluster point y_0. Then $(\alpha, y_0) \in \mathrm{Cl}\, C$. Hence φ is a quasi-perfect map since $\varphi^{-1}(\alpha) = \{\alpha\} \times \left[\bigcap_{i=1}^{\infty} F_{i\alpha_i}\right]$ is countably compact for each $\alpha \in B$. Thus X is an M-space.

Now define a map $f : X \to Y$ by $f(\alpha, y) = y$. Take $A \subset X$; assume $f(A)$ not closed. By hypothesis (a), there exists a countably compact set $K \subset Y$ such that $f(A) \cap K$ is not closed in K. So there exists a point $y_0 \in K$ such that $y_0 \in \mathrm{Cl}\, [f(A) \cap K] - f(A)$.

Now, $A = \cup \{A \cap [B(\lambda) \times Y] : \lambda \in \Omega_1\}$, since $B = \cup [B(\lambda) : \lambda \in \Omega_1]$ and $A \subset X \subset B \times Y$. Then we have

$$f(A) = \cup \{f(A \cap B(\lambda) \times Y) : \lambda \in \Omega_1\},$$
$$f(A) \cap K = \cup \{f(A \cap B(\lambda) \times Y) \cap K : \lambda \in \Omega_1\}.$$

Note that $f(A \cap B(\lambda) \times Y) \subset F_{1\lambda}$ and $\{F_{1\lambda} : \lambda \in \Omega_1\}$ is hereditarily closure preserving. So there exists an $\alpha_1 \in \Omega_1$ such that

$$y_0 \in \mathrm{Cl}\, [f(A \cap B(\alpha_1) \times Y) \cap K].$$

Assume that there exists a $k \in N$, $\alpha_k \in \Omega_k$ such that $y_0 \in \mathrm{Cl}\, \{f[A \cap B(\alpha_1, \cdots, \alpha_k) \times Y] \cap K\}$. Call

$$f[A \cap B(\alpha_1, \cdots, \alpha_k) \times Y] = C_k(\alpha_1, \cdots, \alpha_k),$$
$$f[A \cap B(\alpha_1, \cdots, \alpha_k, \lambda) \times Y] = C_k(\alpha_1, \cdots, \alpha_k, \lambda).$$

Note $C_k(\alpha_1, \cdots, \alpha_k) = \cup \{C_k(\alpha_1, \cdots, \alpha_k, \lambda) : \lambda \in \Omega_{k+1}\}$, $C_k(\alpha_1, \cdots, \alpha_k, \lambda) \subset F_{(k+1)\lambda}$ and $\{F_{(k+1)\lambda} : \lambda \in \Omega_{k+1}\}$ is hereditarily closure preserving. Now

$$C_k(\alpha_1, \cdots, \alpha_k) \cap K = \cup \{C_k(\alpha_1, \cdots, \alpha_k, \lambda) \cap K : \lambda \in \Omega_{k+1}\}.$$

Hence there exists an α_{k+1} such that

$$y_0 \in \mathrm{Cl}\, C_{k+1}(\alpha_1, \cdots, \alpha_{k+1}) \cap K$$
$$= \mathrm{Cl}\, \{f[A \cap B(\alpha_1, \cdots, \alpha_k, \alpha_{k+1}) \times Y] \cap K\}.$$

Thus, by induction, for any $n \in N$,

$$y_0 \in \mathrm{Cl}\, (C_n(\alpha_1, \cdots, \alpha_n) \cap K).$$

Next note that $C_n(\alpha_1, \cdots, \alpha_n) \subset \bigcap_{i=1}^{n} F_{i\alpha_i} \subset F_{n\alpha_n}$. If we can show that $\{F_{i\alpha_i} : i \in N\}$ is not hereditarily closure preserving, it will then form a q-sequence at y_0. Now let us abbreviate $C_k(\alpha_1, \cdots, \alpha_k)$ to $C_k(\alpha)$, where $\alpha = (\alpha_1, \alpha_2, \cdots)$. Since $y_0 \in \mathrm{Cl}\,(C_1(\alpha) \cap K)$, there exists y_1 such that $y_1 \in C_1(\alpha) \cap K$. Further, $y_1 \neq y_0$ since $y_0 \notin f(A)$.

The space Y is T_1, so a neighborhood $V_1(y_0)$ of y_0 exists such that $y_1 \notin V_1(y_0)$. Now $y_0 \in \mathrm{Cl}\,[C_2(\alpha) \cap K] \cap V_1(y_0) \subset \mathrm{Cl}\,[C_2(\alpha) \cap K \cap V_1(y_0)]$. So there exists $y_2 \neq y_0$ such that $y_2 \in C_2(\alpha) \cap K \cap V_1(y_0)$. Then $V_2(y_0)$ exists such that $y_2 \notin V_2(y_0)$, $V_2(y_0) \subset V_1(y_0)$. By induction, there exists $y_n \in C_n(\alpha) \cap K \cap V_{n-1}(y_0)$ such that $y_n \neq y_0$, $y_n \notin V_n(y_0)$, $V_n(y_0) \subset V_{n-1}(y_0)$.

If some y_k is a cluster point of the sequence $\{y_n\}$, then y_k is also a cluster point of the sequence $\{y_n : n > k\}$ and hence $\{F_{n\alpha_n} : n > k\}$ is not hereditarily closure preserving. If no y_n is a cluster point of the sequence $\{y_n\}$, $\{F_{i\alpha_i} : i \in N\}$ is not hereditarily closure preserving, since $\{y_n\} \subset K$. Thus in any case $\{F_{i\alpha_i} : i \in N\}$ is not hereditarily closure preserving, so it forms a q-sequence at y_0,

$$y_0 \in \bigcap_{n=1}^{\infty} \mathrm{Cl}\,(C_n(\alpha) \cap K) \subset \bigcap_{i=1}^{\infty} F_{i\alpha_i}, \quad \text{and} \quad \alpha \in B.$$

We shall now prove that $(\alpha, y_0) \in \mathrm{Cl}\, A$. Any neighborhood of (α, y_0) has the form

$$[B(\alpha_1, \cdots, \alpha_n) \times V(y_0)] \cap X$$

where $V(y_0)$ is a neighborhood of y_0 in Y. Note that

$$V(y_0) \cap C_n(\alpha) \neq \emptyset \qquad \text{for every } n \in N,$$

so $\emptyset \neq f^{-1}(V(y_0)) \cap [A \cap B(\alpha_1, \cdots, \alpha_n) \times Y] \subset [B(\alpha_1, \cdots, \alpha_n) \times V(y_0)] \cap A$. The fact that the latter set is not empty implies that $(\alpha, y_0) \in \mathrm{Cl}\, A$. Since $f(\alpha, y_0) = y_0$ and $y_0 \notin f(A)$, $(\alpha, y_0) \notin A$. Thus $f(A)$ nonclosed implies A nonclosed. So the map f is closed.

(ii) **Proof of the "only if" part.**

Let X be M; $f : X \to Y$ a closed map. Nagata [5] has shown that (a) holds.

So let $\{\mathcal{U}_n\}$ be a normal sequence of locally finite open covers of X satisfying (1) of Definition 2.4. Put $\mathcal{F}_n = \{\mathrm{Cl}\, U : U \in \mathcal{U}_n\}$. Then it is easy to see that $\{f(F_n)\}$ forms a family of hereditarily closure preserving closed covers of Y. It remains to show that $\{f(\mathcal{F}_n)\}$ forms an almost q-refining sequence.

Suppose $y \in f(A_n)$, $A_n \in \mathcal{F}_n$, $n \in N$, and that $\{f(A_n)\}$ is not hereditarily closure preserving. Then $\{A_n\}$ is not hereditarily closure preserving. Hence a family $\{K_n\}$ of closed sets exists such that

$$K_n \subset A_n,\ \mathrm{Cl}\left(\bigcup_{n=1}^{\infty} K_n\right) - \bigcup_{n=1}^{\infty} K_n \neq \emptyset.$$

Let $x_0 \in \mathrm{Cl}\left(\bigcup_{n=1}^{\infty} K_n\right) - \bigcup_{n=1}^{\infty} K_n$. Then $\mathrm{St}(x_0, \mathcal{U}_i)$ intersects infinitely many K_n, since otherwise we would have

$$x_0 \in \mathrm{Cl}\left(\bigcup_{i=1}^{m} K_i\right) = \bigcup_{i=1}^{m} K_i \qquad \text{for some } m \in N.$$

Hence an increasing sequence $\{n_i\}$ of natural numbers exists such that

$$\mathrm{St}(x_0, \mathcal{U}_i) \cap K_{n_i} \neq \emptyset, \qquad i \in N.$$

Let $x_n \in A_n = \mathrm{Cl}\, U_n$ for $U_n \in \mathcal{U}_n$ and $n \in N$. Since $K_{n_i} \subset A_{n_i}$, we have $\mathrm{St}(x_0, \mathcal{U}_i) \cap \mathrm{Cl}\, U_{n_i} \supset \mathrm{St}(x_0, \mathcal{U}_i) \cap K_{n_i} \neq \emptyset$. Thus $\mathrm{St}(x_0, \mathcal{U}_i) \cap U_{n_i} \neq \emptyset$ and hence

$$U_{n_i} \subset \mathrm{St}(\mathrm{St}(x_0, \mathcal{U}_i), \mathcal{U}_{n_i}) \subset \mathrm{St}(\mathrm{St}(x_0, \mathcal{U}_i), \mathcal{U}_i) \subset \mathrm{St}(x_0, \mathcal{U}_{i-1}).$$

Then $x_{n_i} \in \mathrm{Cl}\, U_{n_i} \subset \mathrm{Cl}(\mathrm{St}(x_0, \mathcal{U}_{i-1})) \subset \mathrm{St}(x_0, \mathcal{U}_{i-2})$. So $\{x_{n_i}\}$ clusters, and hence $\{x_n\}$ has a cluster point. Now let $y_n \in f(A_n)$, $n \in N$. Then there are $x_n \in A_n$ such that $y_n = f(x_n)$. Since $\{x_n\}$ has a cluster point in X, $\{y_n\}$ has a cluster point in Y. Thus $\{f(A_n)\}$ forms an almost q-refining sequence.

We have shown that if $\{A_n\}$ is a q-sequence at x, then $\{f(A_n)\}$ is a q-sequence at $f(x)$, so (b) is proved.

This completes the proof of our characterization theorem.

Remark. Theorem 3.1 remains true if we replace (a) by the property described in Rishel [6, Theorem 1]; because a regular space with this property is a quotient image of a regular M-space by [6] and hence quasi-k by [5].

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