223. *On an Extension Theorem for Turning Point Problem*

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§1. Introduction. Let us consider the asymptotic properties of solutions of differential equations depending on a small positive parameter ε when ε tends to zero. If there are some properties which do not stand uniformly for the region considered that is, there exist points in the region which we call singular points where the asymptotic nature breaks down, the studies of such phenomenon are often called the singular perturbation problems. Many important problems from the applied mathematics and theoretical physics can be converted to the research of these problems. The boundary layer theory of fluid mechanics, the relaxation oscillation in the circuit theory, and the turning point problems in quantum mechanics are familiar among others.

In this paper, we restrict ourselves to consider the asymptotic expansions of solutions of the second order ordinary differential equations containing turning points. At the turning point, an asymptotic expansion in power series of ε breaks down, and if we want to have uniformly valid asymptotic expansion in the region containing turning points, it is necessary to consider much more complicated series of ε than integral power series of ε.

Here we talk about the stretching and matching method used frequently in fluid mechanics. This consists of the following three procedures, at first an asymptotic expansion of solution which is conveniently called outer solution is calculated in some region where no singular point contains, nextly an asymptotic expansion called inner solution is obtained in a small neighborhood of singular point by appropriate stretching transformations of independent variable, and thereafter the relation between the outer solution and the inner solution is considered. To perform these procedures rigorously, the two regions of existence of the outer and inner solutions must overlap with each other for all sufficiently small ε. For this purpose the following Kaplun's Extension theorem is an essential tool and its proof is given, for example, in Eckhaus [2].

Kaplun's extension theorem.

(i) *Let* $F(x,\varepsilon)$ *be defined for* (x,ε) *in* $[0<x\leqq x_0, 0<\varepsilon\leqq\varepsilon_0]$ *and* $\lim_{\varepsilon\to 0} F(x,\varepsilon)=0$, uniformly in $x_1\leqq x\leqq x_0$ for all $x_1>0$.

Then there exists $\delta_1(\varepsilon)>0$ *such that*

$$\lim_{\varepsilon\to 0} F(x,\varepsilon)=0, \text{ uniformly in } \delta_1(\varepsilon)\leqq x\leqq x_0 \text{ with } \lim_{\varepsilon\to 0}\delta_1(\varepsilon)=0.$$

(ii) *Let* $F(x,\varepsilon)$ *be defined for* (x,ε) *in* $[x_0\leqq x<\infty, 0<\varepsilon\leqq\varepsilon_0]$ *and*

$$\lim_{\varepsilon\to 0} F(x,\varepsilon)=0, \text{ uniformly in } x_0\leqq x\leqq R \text{ for all } R\geqq x_0.$$

Then there exists $\delta_2(\varepsilon)>0$ *such that*

$$\lim_{\varepsilon\to 0} F(x,\varepsilon)=0, \text{ uniformly in } x_0\leqq x\leqq \delta_2(\varepsilon)^{-1} \text{ with } \lim_{\varepsilon\to 0}\delta_2(\varepsilon)=0.$$

It is in general very difficult to obtain the functions $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ explicitly and even to prove that the two regions of existence overlap with each other for the given differential equations. In spite of some ambiguities in these respects, this stretching and matching method is important and used in many articles and books: Fraenkel [4], Cole [1], Lagerstrom, Howard and Liu [5] and Van Dyke [7].

The purpose of this paper is to extend the region of existence of asymptotic expansion of solutions of second order ordinary differential equations containing turning points, that is to get $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ explicitly in Kaplun's extension theorem. The results presented here play essential role when we analyze the central connection problem at a turning point by the matching method under the assumption that the characteristic polygon associated with this turning point consists of two segments.

§2. The domain of influence and the canonical region.

We consider in this paper the differential equation in vector form

$$(2.1)\qquad \varepsilon^h\frac{dy}{dx}=\begin{bmatrix}0 & 1\\ p(x,\varepsilon), & 0\end{bmatrix}y \qquad \text{in } D=\{|x|<\infty, |x^\alpha\varepsilon|\leqq\delta_0\},\ 0<\varepsilon\leqq\varepsilon_0,$$

where h is a positive integer, y is 2-vector and x is the complex independent variable, δ_0 and ε_0 are sufficiently small positive constants, and we assume here that the function $p(x,\varepsilon)$ has an asymptotic expansion in power series of ε such that for every m there exists a positive constant M and

$$\left|p(x,\varepsilon)-\sum_{\nu=0}^{m}p_\nu(x)\varepsilon^\nu\right|\leqq M(1+|x|^{(m+1)\alpha+\beta})\varepsilon^{m+1},$$

where $p_\nu(x)$ are polynomials of x of degree at most $\nu\alpha+\beta$ for $\nu\geqq 1$. Here α and β are some positive rational numbers which may be zero. In particular

$$p_0(x)=x^q+p_{0,q-1}x^{q-1}+\cdots+p_{0,0}.$$

We call the zeros of the polynomial $p_0(x)$ the turning points of the equation (2.1), and the order of zero is said the order of the turning point. We denote in this paper the turning point by a_i. To each turning point a_i it corresponds a rational number ρ_{a_i} which is determined from the characteristic polygon Γ_{a_i} associated with a_i, that is, $\rho_{a_i}=-\mu_{a_i}^{-1}$, here μ_{a_i} denotes the tangent of the segment of Γ_{a_i} which is situated at the highest position. For the characteristic polygon of second order equations we refer for example to Nakano and Nishimoto [6].

The equation (2.1) is changed by the transformation of the form

$$y=\begin{bmatrix} 1 & 1 \\ \sqrt{p_0} & -\sqrt{p_0} \end{bmatrix}(E+\varepsilon Q_1)(E+\varepsilon^2 Q_2)\cdots(E+\varepsilon^{m+h}Q_{m+h})z_m$$

into

$$\varepsilon^h \frac{dz_m}{dx}=\sqrt{p_0}\left\{\sum_{\nu=0}^{m+h} G_\nu(x)\varepsilon^\nu + R_{m+h+1}(x,\varepsilon)\varepsilon^{m+h+1}\right\} z_m. \tag{2.2}$$

Here $G_\nu(x)$ and $Q_\nu(x)$ are diagonal and antidiagonal two-to-two matrices respectively, and their elements are determined successively from the elements of $Q_j(x)(j<\nu)$ and $p_\mu(x)(\mu\leqq\nu)$. For the matrices $G_\nu(x)$, $Q_\nu(x)$ and the remainder term $R_{m+h+1}(x,\varepsilon)$, we can prove the following lemmas.

Lemma 2.1. *The growth order of elements of the matrices $G_\nu(x)$ and $Q_\nu(x)$ as x tends to infinity is at most $\nu(\alpha+\beta-q)$ if $\beta\geqq q$, and $\nu\alpha+\beta-q$ if $\beta<q$. The order of pole of them at a turning point a_i is at most ν/ρ_{a_i}.*

Lemma 2.2. *For the remainder term $\varepsilon^{m+h+1}\sqrt{p_0(x)}R_{m+h+1}(x,\varepsilon)$, we have*

$$\varepsilon^{m+h+1}\sqrt{p_0}R_{m+h+1}=\begin{cases} 0[(x^{\alpha+\beta-q}\varepsilon)^{m+h+1}\cdot x^{q/2}] & \text{if } \beta\geqq q, \\ 0[(x^{\alpha}\varepsilon)^{m+h+1}\cdot x^{\beta-q/2}] & \text{if } \beta<q \end{cases}$$

as x tends to infinity under the restriction $|x^{\alpha_1}\varepsilon|\ll 1$, here α_1 denotes the number $\alpha+\beta-q$ if $\beta\geqq q$, and α if $\beta<q$. And we have

$$\varepsilon^{m+h+1}\sqrt{p_0}R_{m+h+1}=0[(|x-a_i|^{-1/\rho_{a_i}}\varepsilon)^{m+h+1}\cdot|x-a_i|^{r_i/2}]$$

as x tends to a turning point a_i of order r_i under the restriction $|(x-a_i)^{-1/\rho_{a_i}}\varepsilon|\ll 1$.

Now we introduce the following notations for convenience:

$$\frac{\sqrt{p_0}}{\varepsilon^h}\left\{\sum_{\nu=0}^{h} G_\nu(x)\varepsilon^\nu\right\}=\begin{bmatrix}\gamma_h(x,\varepsilon), & 0 \\ 0 & -\gamma_h(x,\varepsilon)\end{bmatrix}-\frac{p_0'}{4p_0}\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix},$$

$$\gamma_h(x,\varepsilon)=\frac{\sqrt{p_0}}{\varepsilon^h}\left\{1+\frac{p_1}{2p_0}\varepsilon+\left(\frac{p_2}{2p_0}-\frac{1}{8}\left(\frac{p_1}{p_0}\right)^2\right)\varepsilon^2+\cdots+\frac{p_h}{2p_0}\varepsilon^h\right\},$$

$$\xi_0(x,x_0)=\int_{x_0}^{x}\sqrt{p_0(x)}dx, \quad \xi_h(x,x_0,\varepsilon)=\int_{x_0}^{x}\varepsilon^h\gamma_h(x,\varepsilon)dx,$$

$$\wedge_h(x,x_0,\varepsilon)=\varepsilon^{-h}\begin{bmatrix}\xi_h(x,x_0,\varepsilon), & 0 \\ 0 & , -\xi_h(x,x_0,\varepsilon)\end{bmatrix},$$

$$z_m(x,\varepsilon)=\hat{z}_m(x,\varepsilon)p_0^{-1/4}\exp\wedge_h(x,x_0,\varepsilon),$$

$$w_m(x,\varepsilon)=\hat{w}_m(x,\varepsilon)p_0^{-1/4}\exp\wedge_h(x,x_0,\varepsilon),$$

$$\hat{w}_m(x,\varepsilon)=\exp\int^x\left(\sum_{j=1}^{m}\sqrt{p_0(x)}G_{h+j}(x)\varepsilon^j\right)dx.$$

We want to prove that $w_m(x,\varepsilon)$ is an asymptotic approximation of $z_m(x,\varepsilon)$ to the extent of ε^m in a certain region which is wider than the one established earlier. To do this, it will be introduced at first the notion of the domain of influence at a turning point.

For each turning point a_i of order r_i, the domain of influence N_{a_i} is defined by

$$N_{a_i}: |x-a_i| \leqq N\varepsilon^{\lambda_{a_i}}, \qquad \lambda_{a_i} = \left(\frac{h+1}{\rho_{a_i}} - \frac{r_i}{2} - 1\right)^{-1}$$

where N is an appropriately chosen positive constant.

Next, we explain the canonical regions with respect to $\xi_0(x, a)$ which was firstly appeared in the article Evgrafov and Fedoryuk [3]. The family of curves $\mathrm{Re}\,\xi_0(x, a) = \text{const.}$ does not depend on the initial value a, the choice of the paths in the complex x-plane or the determination of the square root $\sqrt{p_0(x)}$, and has branch points at turning points. The curves passing through turning points are called the Stokes curves. These curves divide the x-plane into a finite number of simply connected unbounded regions, which we call Stokes regions. Here we consider the function $\xi_0(x, a)$ as the mapping of the x-plane into the ξ-plane. The canonical region $D[\xi_0]$ with respect to $\xi_0(x, a)$ is a union of an appropriate number of adjacent Stokes regions bounded by the Stokes curves and is mapped by $\xi_0(x, a)$ onto the whole ξ-plane cut by a finite number of unbounded verticals. For each x in $D[\xi_0]$, we can describe smooth curves α_x^+ and α_x^- which start from x and tend to infinity such that $\mathrm{Re}\,\xi_0(x, a)$ is monotone increasing along α_α^+ and monotone decreasing along α_x^-, and $\lim_{x\to\infty} \mathrm{Re}\,\xi_0(x, a) = \pm\infty$.

Let X be the complex x-plane, $D[\xi_0]$ be a canonical region with respect to $\xi_0(x, a)$ and $A = \{a_i\}$ be a set of turning points that are at the boundary of $D[\xi_0]$. Here we define a few types of region successively which depend on ε.

$$D[\xi_0, N] = D[\xi_0] \cap \{X - \bigcup_{\alpha_i \in A} Na_i\}$$

where Na_i is the domain of influence at the turning point a_i.

$$D[\xi_0, N, \delta_2, \varepsilon] = D[\xi_0, N] \cap \{x : |x^{\alpha_2}\varepsilon| \leqq \delta_2\} \qquad \text{for } 0 < \varepsilon \leqq \varepsilon_0,$$

where δ_2 is a sufficiently small positive constant and α_2 is a positive rational number defined by $(h+1)(\alpha+\beta-q)+q/2+1$ if $\beta \geqq q$, $(h+1)\alpha + \beta - q/2 + 1$ if $q/2 - 1 - \alpha h \leqq \beta < q$, and α if $\beta < q/2 - 1 - \alpha h$.

By using these regions the following lemma can be proved.

Lemma 2.3. *There exists a region $D[\xi_h, N, \delta_2, \varepsilon]$ which we call canonical region with respect to $\xi_h(x, a, \varepsilon)$ and is obtained by a slight deformation of $D[\xi_0, N, \delta_2, \varepsilon]$ with the conditions: for each $x \in D[\xi_h, N, \delta_2, \varepsilon]$, there exist two points $x_0^+(\varepsilon)$, $x_0^-(\varepsilon)$ and two curves $\alpha^+(s, x, x_0^+)$, $\alpha^-(s, x, x_0^-)$ defined for $0 \leqq s \leqq s_{x_0^\pm}$ and connecting x and $x_0^+(\varepsilon)$, $x_0^-(\varepsilon)$ respectively such that*

(1) *$x_0^\pm(\varepsilon)$ are on the boundary of $D[\xi_h, N, \delta_2, \varepsilon]$ and $\alpha^\pm(s, x, x_0^\pm)$ are contained in $D[\xi_h, N, \delta_2, \varepsilon]$,*

(2) $\alpha^{\pm}(0, x, x_0^{\pm}) = x$, $\alpha^{\pm}(s_{x_0^{\pm}}, x, x_0^{\pm}) = x_0^{\pm}(\varepsilon)$,

(3) $\mathrm{Re}\,\xi_h(x, a, \varepsilon)$ *is monotone increasing along* $\alpha^+(s, x, x_0^+)$, *monotone decreasing along* $\alpha^-(s, x, x_0^-)$, *and* $\lim \mathrm{Re}\,\xi_h(x, a, \varepsilon) = \pm\infty$ *as* x *tends to infinity along* $\alpha^{\pm}(s, x, x_0^{\pm})$.

§ 3. Existence theorem.

To state the main theorem precisely, it is convenient to divide the canonical region $D[\xi_h, N, \delta_2, \varepsilon]$ into a finite number of subregions such that

$$D^{(a_i)}[\xi_h, N, \delta_2, \varepsilon] = D[\xi_h, N, \delta_2, \varepsilon] \cap \{x : N\varepsilon^{\lambda a_i} \leqq |x - a_i| \leqq \delta_3\},$$

for $a_i \in A$, $0 < \varepsilon \leqq \varepsilon_0$, where δ_3 and ε_0 are taken sufficiently small.

$$D^{(\infty)}[\xi_h, N, \delta_2, \varepsilon] = D[\xi_h, N, \delta_2, \varepsilon] - \bigcup_{a_i \in A} D^{(a_i)}[\xi_h, N, \delta_2, \varepsilon].$$

Now we can prove the main theorem by using the Lemmas in § 2.

Theorem 3.1. *There exists a fundamental system of solutions* $z_m(x, \varepsilon)$ *of the equation* (2.2) *such that*

$$\|\{z_m(x, \varepsilon) - w_m(x, \varepsilon)\} p_0^{1/4} \wedge_h^{-1}(x, x_0,)\| \leqq \begin{cases} K[1 + |x|^{(m+1)\alpha_2}]\varepsilon^{m+1} & \text{for } x \in D^{(\infty)}[\xi_h, N, \delta_2, \varepsilon], \\ K[|x - a_i|^{-(m+1)/\lambda a_i}]\varepsilon^{m+1} & \text{for } x \in D^{(a_i)}[\xi_h, N, \delta_2, \varepsilon]. \end{cases}$$

Theorem 3.2. *The differential equation* (2.1) *has a fundamental system of solutions of the form*

$$y(x, \varepsilon) = \begin{bmatrix} 1 & 1 \\ \sqrt{p_0} & -\sqrt{p_0} \end{bmatrix} p_0^{-1/4}\{E + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots + \varepsilon^m y_m(x) + Y_{m+1}(x, \varepsilon)\} \exp \wedge_h(x, x_0, \varepsilon),$$

where E *is the two-to-two unit matrix and the remainder term* $Y_{m+1}(x, \varepsilon)$ *satisfies*

$$\|Y_{m+1}(x, \varepsilon)\| \leqq \begin{cases} K[1 + |x|^{(m+1)\alpha_2}]\varepsilon^{m+1} & \text{for } x \in D^{(\infty)}[\xi_h, N, \delta_2, \varepsilon], \\ K[|x - a_i|^{-1/\lambda a_i}\varepsilon]^{m+1} & \text{for } x \in D^{(a_i)}[\xi_h, N, \delta_2, \varepsilon]. \end{cases}$$

From these results we can see that the functions $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ in the Kaplun's extension theorem are in our case $N\varepsilon^{\lambda a_i}$ and $[\delta_2^{-1}\varepsilon]^{1/\alpha_2}$ respectively.

The detailed descriptions of the proof of our lemmas and theorems, and their applications will be given elsewhere.

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