222. Euclidean Space Bundles and Disk Bundles

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0. Introduction.

A useful property of the orthogonal group $O(n)$ is that it leaves invariant the unit disk D^n and unit sphere S^{n-1} of $\boldsymbol{R}^n$. Consequently one may pass freely from $\boldsymbol{R}^n$-bundles to D^n- and S^{n-1}-bundles in the case where the structure group is $O(n)$. This convenient coincidence does not occur in the topological category. W. Browder [2] showed that some $\boldsymbol{R}^n$-bundles do not contain any D^n-subbundles.

In this paper we shall study on the relationship between $\boldsymbol{R}^n$-bundles and D^n-bundles.

The main result of this paper is the following

Theorem 1. *Let K be a locally finite simplicial complex of dimension k, and $k<n-3$ and $n\geqq 6$. Then the set of all equivalence classes of D^n-bundles over K is canonically in one-to-one correspondence with the set of all equivalence classes of $\boldsymbol{R}^n$-bundles over K.*

In § 1 we prepare on notations and terminologies used later. In § 2 we shall show the stability theorem of the homotopy groups $\pi_k(\mathcal{H}_0(n))$. Here we use the recent result of R. Kirby and L. Siebenmann [4]. In § 3 we shall prove the theorem 1.

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1. Notations and terminologies.

Let $\mathcal{H}_0(n)$ be the space of all homeomorphisms of the Euclidean n-space $\boldsymbol{R}^n$ onto itself preserving the origin 0 with compact-open topology. Then $\mathcal{H}_0(n)$ forms a topological group with the composition of maps (cf. Kister [5]).

By an *$\boldsymbol{R}^n$-bundle* we shall mean a fibre bundle whose fibre is the Euclidean n-space $\boldsymbol{R}^n$ and structure group $\mathcal{H}_0(n)$.

Let $B_{\mathcal{H}_0(n)}$ be the classifying space for the topological group $\mathcal{H}_0(n)$. Its existence is assured by J. Milnor [6]. Then for a finite complex K, the set $[K, B_{\mathcal{H}_0(n)}]$ of all homotopy classes of continuous maps of K into $B_{\mathcal{H}_0(n)}$ is in one-to-one correspondence with the set of all equivalence classes of $\boldsymbol{R}^n$-bundles over K.

On the other hand, we shall denote by TOP_n the *css*-group of all isomorphism-germs of trivial microbundles over simplexes (for the precise definition, see [1]; where we write H_n for TOP_n). Let B_{TOP_n} be the classifying *css*-complex for *css*-group TOP_n. Then for a finite complex K, the set $[\tilde{K}, B_{TOP_n}]$ of all *css*-homotopy classes of *css*-maps of *css*-complex $\tilde{K}$ into B_{TOP_n} is in one-to-one correspondence with the set of all isomorphism classes of microbundles of dimension n over K, where $\tilde{K}$ is the *css*-complex corresponding to K (cf. [1]).

According to J. Kister [5], any microbundle of dimension n over a finite complex contains an $\boldsymbol{R}^n$-bundle, unique up to isomorphism. Therefore, we have the following

Proposition. *We have*

$$\pi_k(\mathcal{H}_0(n)) \cong \pi_k(TOP_n).$$

Let $\mathcal{H}_0(D^n)$ be the space of all homeomorphisms of n-disk D^n onto itself preserving the origin 0 with the compact-open topology. Then $\mathcal{H}_0(D^n)$ can canonically be considered as a subspace of $\mathcal{H}_0(n)$. It follows that $\mathcal{H}_0(D^n)$ also is a topological group.

By a *D^n-bundle* we shall mean a fibre bundle whose fibre is the n-disk D^n and structure group $\mathcal{H}_0(D^n)$.

2. Stability theorem for the homotopy groups $\pi_k(TOP_n) = \pi_k(\mathcal{H}_0(n))$.

We have the canonical inclusion $\iota_n : TOP_n \to TOP_{n+1}$. Then ι_n induces the homomorphism

$$(\iota_n)_* : \pi_k(TOP_n) \to \pi_k(TOP_{n+1}).$$

In this section we shall show the stability theorem for the homotopy groups $\pi_k(TOP_n)$.

Let PL_n be the *css*-group of all isomorphism-germs of PL-microbundles of dimension n over simplexes (cf. Milnor [7]). Then we have the canonical inclusion $\iota_n : PL_n \to PL_{n+1}$. A PL-microbundle can be considered to be a microbundle. Therefore, we have following *css*-homomorphism $\rho_n : PL_n \to TOP_n$. A. Haefliger and C. Wall [3] proved the following stability theorem.

Theorem (Haefliger and Wall). *We have*

$$\pi_k(PL_{n+1}, PL_n) = 0 \qquad \textit{for } k < n.$$

We have the natural *css*-map $TOP_n/PL_n \to TOP_{n+1}/PL_{n+1}$. Recently R. Kirby and L. Siebenmann [4] have proved the following stability.

Theorem (Kirby and Siebenmann).

$$\pi_k(TOP_n/PL_n) \cong \pi_k(TOP_{n+1}/PL_{n+1}), \qquad \textit{for } k < n, \quad n \geqq 5.$$

From these two theorems we have the stability theorem for $\pi_k(TOP_n)$.

Theorem 2. *We have*

$$\pi_k(TOP_{n+1}, TOP_n) = 0, \qquad \textit{for } k < n-1, \quad n \geqq 5.$$

Proof. We consider the commutative diagram on the next page. Then Theorem follows from the two theorems stated above by the five lemmas easily.

$$
\begin{array}{ccccccccccc}
 & & & & \vdots & & \vdots & & & & \vdots \\
 & & & & \downarrow & & \downarrow & & & & \downarrow \\
 & & & & \pi_{k+1}(PL_{n+1}, PL_n) & & \pi_{k+1}(TOP_{n+1}, TOP_n) & & & & \pi_k(PL_{n+1}, PL_n) \\
 & & & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots \longrightarrow & \pi_{k+1}(TOP_n/PL_n) & \longrightarrow & & \pi_k(PL_n) & \xrightarrow{(\rho_n)_*} & \pi_k(TOP_n) & \longrightarrow & \pi_k(TOP_n/PL_n) & \longrightarrow & \pi_{k-1}(PL_n) \longrightarrow \cdots \\
 & \downarrow & & & \downarrow (\iota_n)_* & & \downarrow (\iota_n)_* & & \downarrow & & \downarrow \\
\longrightarrow & \pi_{k+1}(TOP_{n+1}/PL_{n+1}) & \longrightarrow & & \pi_k(PL_{n+1}) & \xrightarrow{(\rho_{n+1})_*} & \pi_k(TOP_{n+1}) & \longrightarrow & \pi_k(TOP_{n+1}/PL_{n+1}) & \longrightarrow & \pi_{k-1}(PL_{n+1}) \longrightarrow \cdots \\
 & & & & \downarrow & & \downarrow & & & & \downarrow \\
 & & & & \pi_k(PL_{n+1}, PL_n) & & \pi_k(TOP_{n+1}, TOP_n) & & & & \pi_{k-1}(PL_{n+1}, PL_n) \\
 & & & & \downarrow & & \downarrow & & & & \downarrow \\
 & & & & \vdots & & \vdots & & & & \vdots
\end{array}
$$

3. Euclidean space bundles and disk bundles.

By using the stability theorem for $\pi_k(\mathcal{H}_0(n))$, we shall prove the following theorem.

Theorem 3. *We have*

$$\pi_k(\mathcal{H}_0(n), \mathcal{H}_0(D^n))=0, \qquad \text{for } k<n-2,\ n\geqq 6.$$

Before we prove the theorem, let us prepare some lemmas.

Let $\mathcal{H}(S^n)$ be the space of all homeomorphisms of n-sphere S^n onto itself with the compact-open topology.

There is a natural restriction homomorphism $\lambda : \mathcal{H}_0(D^n)\rightarrow\mathcal{H}(S^{n-1})$. Radial extension defines a continuous homomorphism $\rho : \mathcal{H}(S^{n-1})\rightarrow\mathcal{H}_0(D^n)$, by

$$\rho(f)(x)=\|x\|\, f(x/\|x\|), \qquad \text{for } x\in D^n,\ x\neq 0$$
$$\rho(f)(0)=0.$$

Clearly $\lambda\circ\rho=$identity on $\mathcal{H}(S^{n-1})$.

Thus we can consider $\mathcal{H}(S^{n-1})$ to be a subspace of $\mathcal{H}_0(D^n)$ canonically.

Lemma 1. *$\rho\circ\lambda$ is homotopic to the identity on $\mathcal{H}_0(D^n)$, so that λ and ρ are inverse homotopy equivalences of $\mathcal{H}_0(D^n)$ and $\mathcal{H}(S^{n-1})$.*

Proof. The homotopy is the Alexander construction. Define $H_t : \mathcal{H}_0(D^n)\rightarrow\mathcal{H}_0(D^n)$ by

$$H_t(f)(x)=\begin{cases}\|x\|\, f(x/\|x\|), & \text{if } \|x\|\geqq 1-t,\ t<1, \text{ or } t=1,\ x\neq 0,\\ (1-t)f(x/(1-t)), & \text{if } \|x\|<1-t,\\ 0, & \text{if } t=1,\ x=0.\end{cases}$$

Then H_t is a homotopy of $\rho\circ\lambda$ and the identity.

Let $p,\ q\in S^n$, $p\neq q$. Then we shall denote by $\mathcal{H}_{p,q}(S^n)$ the subspace of $\mathcal{H}(S^n)$ of those elements that preserve $\{p, q\}$ pointwise. Then we have

Lemma 2. *$\mathcal{H}_0(n)$ is homeomorphic to $\mathcal{H}_{p,q}(S^n)$.*

Proof. There is a natural homeomorphism $\varphi : (S^n-q, p)\rightarrow(\boldsymbol{R}^n, 0)$. Let us define $\Phi : \mathcal{H}_{p,q}(S^n)\rightarrow\mathcal{H}_0(n)$, by $\Phi(f)=\varphi\circ f\circ\Phi^{-1}$. Then Φ gives a homeomorphism between $\mathcal{H}_{p,q}(S^n)$ and $\mathcal{H}_0(n)$.

Lemma 3. *Let $j: \mathcal{H}_{p,q}(S^n) \to \mathcal{H}_p(S^n)$ be the inclusion map. Then j is a weak homotopy equivalence.*

Proof. Let $\pi: \mathcal{H}_p(S^n) \to S^n - p$ be the map defined by $\pi(f) = f(q)$. Then $(\mathcal{H}_p(S^n), \pi, S^n - p)$ is a fibre space and its fibre has the same homotopy type as $\mathcal{H}_{p,q}(S^n)$. Thus we have the lemma.

Lemma 4. *Let $i: \mathcal{H}_p(S^n) \to \mathcal{H}(S^n)$ be the inclusion map. Then i is $(n-1)$-connected.*

Proof. Let $\pi: \mathcal{H}(S^n) \to S^n$ be a map defined by $\pi(f) = f(p)$. Then, as is shown in Browder [2], § 5, $(\mathcal{H}(S^n), \pi, S^n)$ is a fibre bundle whose fibre is $\mathcal{H}_p(S^n)$. Thus we have the lemma.

By Lemmas 2, 3, 4, we have the following

Lemma 5. *The composite map*

$$i \circ j \circ \Phi^{-1}: \mathcal{H}_0(n) \to \mathcal{H}(S^n)$$

is $(n-1)$-connected.

Now we prove the theorem. Let us consider the homotopy exact sequence of the tripe $(\mathcal{H}_0(n), \mathcal{H}_0(D^n), \mathcal{H}(S^{n-1}))$:

$$\cdots \xrightarrow{\partial_*} \pi_k(\mathcal{H}_0(D^n), \mathcal{H}(S^{n-1})) \xrightarrow{i_*} \pi_k(\mathcal{H}_0(n), \mathcal{H}(S^{n-1})) \xrightarrow{j_*} \pi_k(\mathcal{H}_0(n), \mathcal{H}_0(D^n))$$
$$\xrightarrow{\partial_*} \pi_{k-1}(\mathcal{H}_0(D^n), \mathcal{H}(S^{n-1})) \xrightarrow{i_*} \cdots .$$

By Lemma 1 we have $\pi_k(\mathcal{H}_0(D^n), \mathcal{H}(S^{n-1})) = 0$, $k > 0$. Therefore, we obtain that $\pi_k(\mathcal{H}_0(n), \mathcal{H}_0(D^n))$ is isomorphic to $\pi_k(\mathcal{H}_0(n), \mathcal{H}(S^{n-1}))$ for any $k > 0$. However, by Lemma 5, $\pi_k(\mathcal{H}_0(n), \mathcal{H}(S^{n-1}))$ is isomorphic to $\pi_k(\mathcal{H}_0(n), \mathcal{H}_0(n-1))$. On the other hand, we know that $\pi_k(\mathcal{H}_0(n), \mathcal{H}_0(n-1)) = 0$, for $k < n-2$, $n \geqq 6$, by Theorem 2 and Proposition. Thus we obtain the theorem.

Theorem 1 follows easily from Theorem 3.

Added in proof: Theorem 1 can be also proved by Theorem 2 and Theorem A in M. Hirsch's paper: Non linear cell bundles. Ann. of Math., Vol. 84, 373–385 (1966).

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